1 4th Year Materials Engineering

Mechanics of Composite Materials – Lecture 4

2 Last Week

2.1 Summary

http://mconry.ucd.ie/ mconry/4th_Materials_Engineering/

- Stiffness c_{ijkl}
- 21 unique values
- Reduced notation C_{IJ}, 6×6 matrix
- Material Symmetries
  - Some terms $\to 0$
  - Some terms functions of other terms
  - $\Rightarrow$ less than 21 indept. vals
- Plane stress

3 Orthotropic – Stiffness Matrix

3.1 3 Perpendicular Planes of Symmetry

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0\\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0\\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & C_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & C_{55} & 0\\ 0 & 0 & 0 & 0 & 0 & C_{66}\end{bmatrix} \begin{bmatrix}\epsilon_{11}\\ \epsilon_{22}\\ \epsilon_{33}\\ \gamma_{23}\\ \gamma_{13}\\ \gamma_{12}\end{bmatrix}$$

Note:

- The planes are aligned along the coordinate axes.
- 9 independent elastic constants

Tensor subscript		**Matrix subscript**
11	$\Rightarrow$	1
22	$\Rightarrow$	2
33	$\Rightarrow$	3
23	$\Rightarrow$	4
13	$\Rightarrow$	5
12	$\Rightarrow$	6

4 Transversely Isotropic – Stiffness Matrix

4.1 Axis of Rotational Symmetry (x_3 axis)

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{55} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(C_{11} - C_{12}) \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{bmatrix}$$

Note:

- The axis is aligned along a coordinate direction
- 5 independent elastic constants

Note, that if you rotated the material so that the axis of symmetry was not aligned along the coordinate direction, then the stiffness matrix would not look as neat. There would not be as many zeros there. However, there would still be only 5 (or 9 for orthotropic) independent numbers. The other terms would be combinations of these numbers.

Also, you should note how the shear and normal components of stress/strain are decoupled. Normal stresses produce only normal strains and vice versa. Equally, shear strain gives rise only to shear stress.

5 Transversely Isotropic – Stiffness Matrix

5.1 Axis of Rotational Symmetry (x_1 axis)

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{12} & C_{23} & C_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(C_{22} - C_{23}) & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{bmatrix}$$

Note:

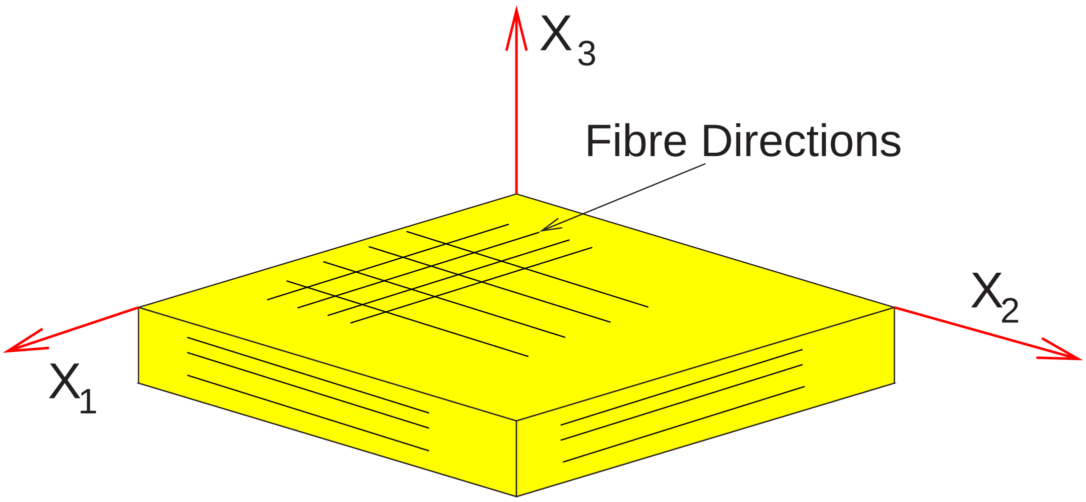
- The axis is aligned along a coordinate direction
- 5 independent elastic constants

6 Plates – Simplifications

6.1 Orthotropic

If laminate is Orthotropic (cross-ply laminate), we get

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{66} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \gamma_{12} \end{bmatrix}$$

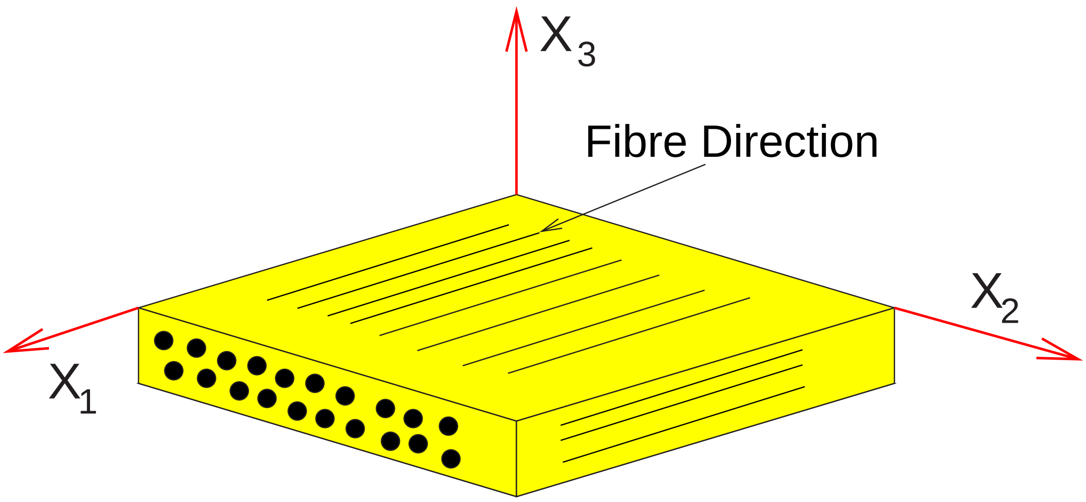


7 Plates – Simplifications

7.1 Transversely Isotropic

If laminate is in fact transversely isotropic (unidirectional laminate), we get

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{66} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \gamma_{12} \end{bmatrix}$$



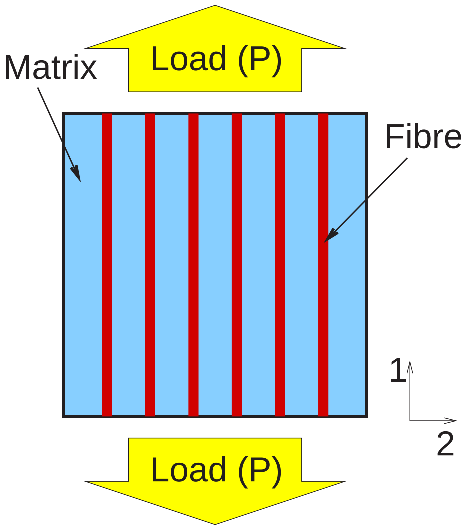
Note that the decoupling of shear/extensional stress/strain is crucial to this step. The plane of the plate has to be closely related to the principle directions of the laminate. Note, corrected from last week.

8 Unidirectional Composite Plate Mechanics

Unidirectional plate loaded parallel to fibres.
Assume perfect fibre–matrix bonding

- $\sigma_f = E_f \epsilon_1$
- $\sigma_m = E_m \epsilon_1$

I.e. they have the same strain, ϵ_1, but (usually) different stresses.



$$\boxed{E_f > E_m \Rightarrow \sigma_f > \sigma_m}$$

9 Unidirectional Composite Plate Mechanics

Say load is P (newtons). Cross Sectional Area is A.
Average stress is

$$\sigma_1 = \frac{P}{A} = \epsilon_1 E_1 \qquad \text{Average for plate}$$

If we are loading parallel to fibres, then

$$\begin{aligned} P &= P_f + P_m \\ &= \sigma_f A_f + \sigma_m A_m \end{aligned}$$

Quantities subscripted with an f refer to the fibres, while subscripts of m refer to the matrix. Divide across by $(A\epsilon_1)$, and recall that $\epsilon_1 = \epsilon_f = \epsilon_m$.

$$\begin{aligned} \frac{P}{\epsilon_1 A} &= \frac{\sigma_f A_f}{\epsilon_1 A} + \frac{\sigma_m A_m}{\epsilon_1 A} \\ E_1 &= E_f \frac{A_f}{A} + E_m \frac{A_m}{A} \end{aligned}$$

10 Unidirectional Composite Plate Mechanics

There is nothing in the composite except fibre and matrix, so this hopefully makes sense.
Note $A_f + A_m = A$ (i.e. there is nothing in the material except fibre and matrix). If the fibres are continuous, then the volume of each phase (fibre, matrix) is proportional to its cross sectional area. This is the same as saying that the volume of a tin can is proportional to the are of the end: volume = area × length. Since both fibre and matrix have the same "length" (the length of the plate along the fibre direction, the ratio of their volumes is the same as the ratio of their cross sectional areas. Volume Fraction of Fibre or Matrix: ϕ_f or ϕ_m.

$$\phi_f = \frac{A_f}{A} \qquad \phi_m = \frac{A_m}{A} \qquad \phi_f + \phi_m = 1$$

Look again at Stiffness of plate:

$$\begin{aligned} E_1 &= E_f \frac{A_f}{A} + E_m \frac{A_m}{A} \\ &= E_f \phi_f + E_m \phi_m \end{aligned}$$

Which we can rewrite:

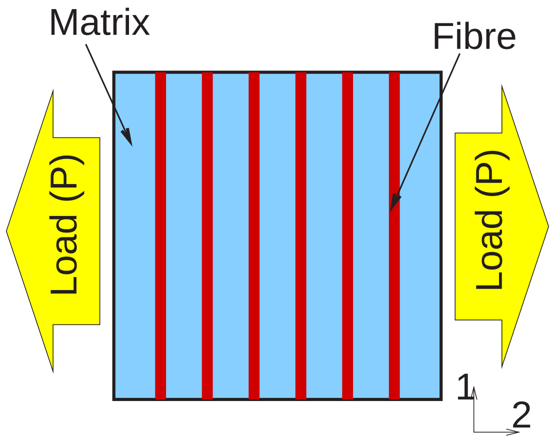
$$E_1 = E_f\phi_f + E_m(1-\phi_f)$$

Rule of Mixtures Equation

11 Unidirectional Composite Plate Mechanics

Now, for the direction normal to the fibre direction. Find E_2. In this case, the applied load acts transverse to the fibres, this means stress on fibre and matrix is the same.

$$\sigma_f = \sigma_m = \sigma_2$$



Strains are:

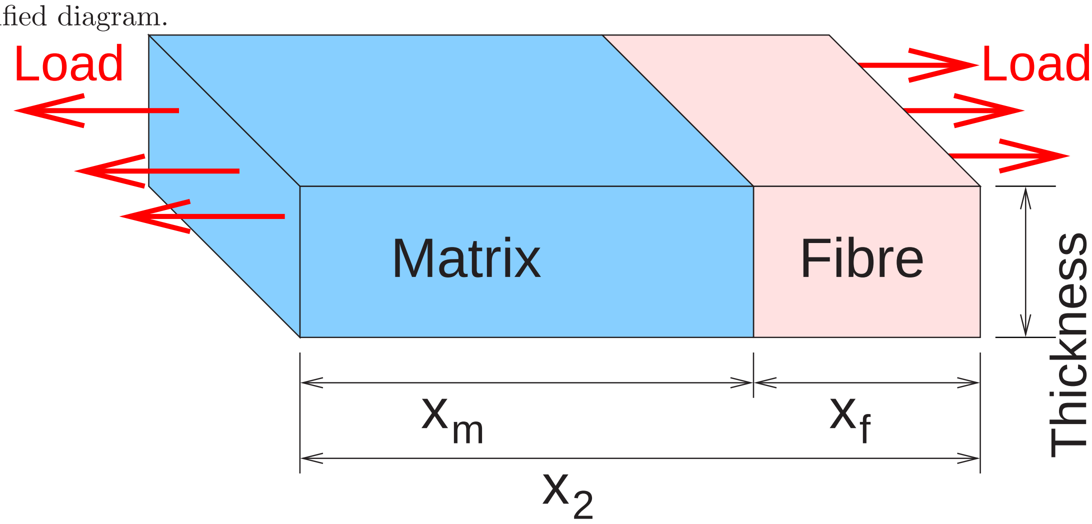
$$\epsilon_f = \sigma_2/E_f \qquad \epsilon_m = \sigma_2/E_m$$

12 Unidirectional Composite Plate Mechanics

The total strain is equal to the average of the strain of the fibres and of the matrix. Average using Volume Fraction.

$$\epsilon_2 = \phi_f\epsilon_f + \phi_m\epsilon_m$$

In case this is not clear, it is explained as follows. Say the Fibre and Matrix are arranged as shown in this simplified diagram.



Clearly, the total original length x_2 is just the sum of the original lengths of the fibres and the matrix.

$$x_2 = x_f + x_m$$

Since the thickness is constant, and again assuming the fibres are continuous (i.e. the the fibre and matrix both extend into the page to the same extent), we can say the following about the volume fraction:

$$\phi_f = \frac{V_f}{V_{total}} = \frac{A_f}{A} = \frac{x_f}{x_2} \quad \Rightarrow \quad x_f = x_2\phi_f$$
$$\phi_m = \frac{V_m}{V_{total}} = \frac{A_m}{A} = \frac{x_m}{x_2} \quad \Rightarrow \quad x_m = x_2\phi_m$$

Now, if we knew the strain of each of fibre and matrix, we could write down the extension of each part (extension is strain times original length). We'll put a Δ in front of a dimension to show that it is an extension. Then we have.

$$\begin{aligned}\Delta x_f &= \epsilon_f x_f = \epsilon_f \phi_f x_2 \\ \Delta x_m &= \epsilon_m x_m = \epsilon_m \phi_m x_2\end{aligned}$$

The extension of the plate as a whole will be the sum of the extensions of the individual phases, which lets us write:

$$\begin{aligned}\Delta x_2 &= \Delta x_f + \Delta x_m \\ &= \epsilon_f \phi_f x_2 + \epsilon_m \phi_m x_2 \\ &= (\epsilon_f \phi_f + \epsilon_m \phi_m)\, x_2\end{aligned}$$

The overall strain ϵ_2 is the overall extension Δx_2 divided by the original length x_2:

$$\begin{aligned}\epsilon_2 &= \frac{\Delta x_2}{x_2} \\ &= \frac{(\epsilon_f \phi_f + \epsilon_m \phi_m)\, x_2}{x_2} \\ &= \epsilon_f \phi_f + \epsilon_m \phi_m\end{aligned}$$

Which is what we set out to show.

Introducing stress (same for fibre and matrix, σ_2):

$$\begin{aligned}\epsilon_2 &= \phi_f \epsilon_f + \phi_m \epsilon_m \\ &= \phi_f \frac{\sigma_2}{E_f} + \phi_m \frac{\sigma_2}{E_m} \\ &= \left(\frac{\phi_f}{E_f} + \frac{\phi_m}{E_m}\right)\sigma_2\end{aligned}$$

Then $$\boxed{E_2 = \frac{\sigma_2}{\epsilon_2} = \left(\frac{\phi_f}{E_f} + \frac{\phi_m}{E_f}\right)^{-1} = \frac{E_f E_m}{E_f \phi_m + E_m \phi_f}}$$

13 Unidirectional Composite Plate Mechanics

Two Equations

$$\boxed{E_1 = E_f \phi_f + E_m(1 - \phi_f) \qquad E_2 = \frac{E_f E_m}{E_f(1 - \phi_f) + E_m \phi_f}}$$

Note: generally $E_f \gg E_m \ldots$

$$\Rightarrow E_1 \approx E_f \phi_f \qquad \text{and} \qquad E_2 \approx \frac{E_m}{(1-\phi_f)}$$

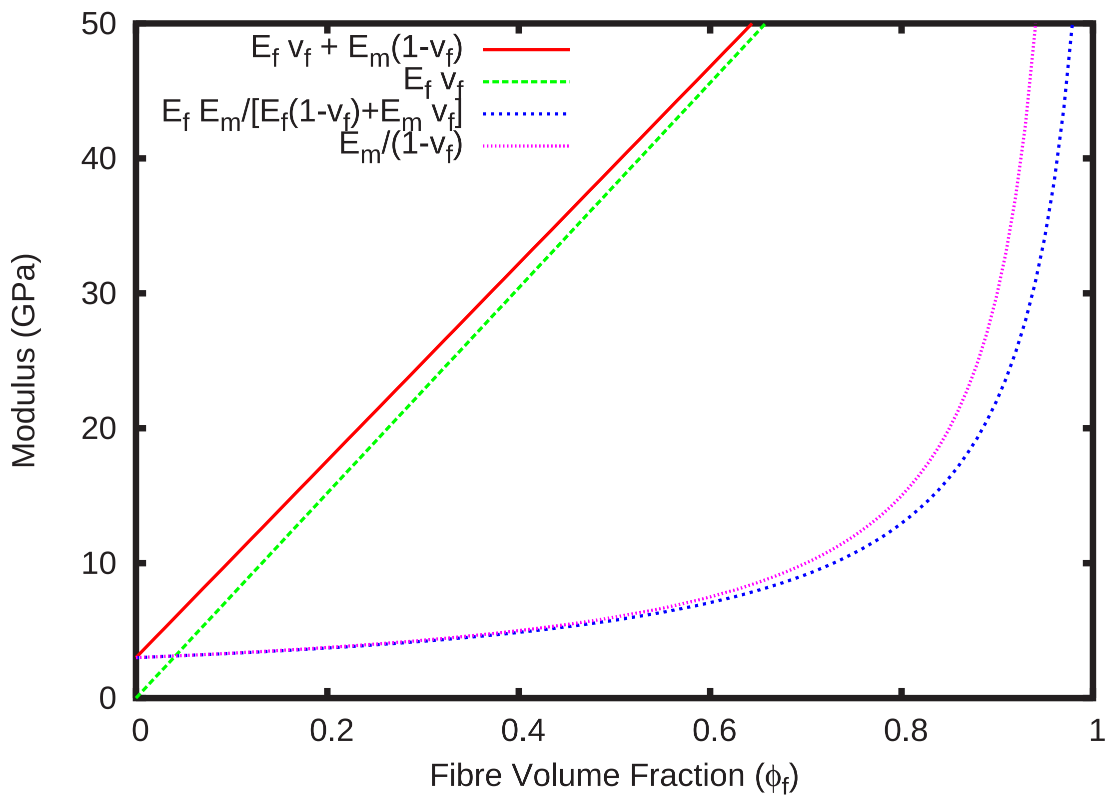
Remember:

- E_1 is stiffness of plate along fibre direction
- E_2 is stiffness of plate parallel to fibre direction

Notice that in the fibre direction, it is primarily the properties and proportion of the fibre that dominate the behaviour

Normal to the fibres, the matrix tends to dominate, though as fibre volume fraction increases the plate becomes stiffer.

14 Unidirectional Composite Plate Mechanics



Note that the properties used here for the fibre and matrix stiffness are the same as used in the worked example later in this document, and relate to glass fibre embedded in a polyester matrix.

15 Unidirectional Composite Plate Mechanics

15.1 Poisson's Ratio ν_{12} (note <u>two</u> values)

ν_{12} is the contraction in direction 2 (normal to fibres), when stress applied to direction 1 (parallel to fibres). $\epsilon_2 = -\nu_{12}\epsilon_1$.

$$\epsilon_1 = \epsilon_{1f} = \epsilon_{1m} \qquad \text{We have seen this before}$$

For each component (fibre, matrix) we can say

$$\epsilon_{2f} = -\nu_f \epsilon_{1f} = -\nu_f \epsilon_1 \qquad \epsilon_{2m} = -\nu_m \epsilon_{1m} = -\nu_m \epsilon_1$$

$$\begin{aligned} \epsilon_2 &= \phi_f \epsilon_{2f} + \phi_m \epsilon_{2m} \qquad \text{We have seen this before too} \\ &= -\phi_f \nu_f \epsilon_1 - \phi_m \nu_m \epsilon_1 \\ &= -(\phi_f \nu_f + \phi_m \nu_m)\epsilon_1 \end{aligned}$$

Therefore, we can say:

$$\nu_{12} = \phi_f \nu_f + \phi_m \nu_m$$

16 Unidirectional Composite Plate Mechanics

16.1 Poisson's Ratio ν_{21} (note two values)

$$\nu_{21} \neq \nu_{12} \qquad \text{N.B. Very Important}$$

If plate is loaded normal to fibres and has strain ϵ_2 in the direction normal to the fibres, what will its strain parallel to the fibres be:

$$\epsilon_1 = -\nu_{21}\epsilon_2$$

ν_{21} is harder to find than ν_{12}. From note (6.N.5) in McCrum and Buckley, the following argument is presented:

First, write down the two strains in detail for a plate loaded parallel to the two axes 1 and 2:

$$\epsilon_1 = \frac{\sigma_1}{E_1} - \nu_{21}\frac{\sigma_2}{E_2} \tag{1}$$

$$\epsilon_2 = -\nu_{12}\frac{\sigma_1}{E_1} + \frac{\sigma_2}{E_2} \tag{2}$$

The strain energy of deformationis given by

$$Q = \frac{1}{2}\left(\sigma_1\epsilon_1 + \sigma_2\epsilon_2\right) \tag{3}$$

Substitute expressions for ϵ_1 and ϵ_2 from above:

$$Q = \frac{1}{2}\left[\frac{\sigma_1^2}{E_1} - \left(\frac{\nu_{21}}{E_2} + \frac{\nu_{12}}{E_1}\right)\sigma_1\sigma_2 + \frac{\sigma_2^2}{E_2}\right] \tag{4}$$

Since we are assuming that the composite is linearly elastic, strain in a direction can be obtained by differentiating Q with respect to stress in that direction.

$$\epsilon_1 = \frac{\partial Q}{\partial \sigma_1} = \frac{\sigma_1}{E_1} - \frac{1}{2}\left(\frac{\nu_{21}}{E_2} + \frac{\nu_{12}}{E_1}\right)\sigma_2 \tag{5}$$

Comparing this with Equation (**??**), and noting that they must agree for any values of stress σ_1 and σ_2, the elastic constants must be related as follows:

$$\frac{\nu_{21}}{E_2} = \frac{\nu_{12}}{E_1}$$

17 Unidirectional Composite Plate Mechanics

17.1 Sample Calculation: Example 6.2

40% ϕ_f, glass fibre in polyester matrix. 100 MPa applied parallel to fibres, predict all the resulting strains. $E_f = 76$GPa, $\nu_f = 0.22$, $E_m = 3$GPa, $\nu_m = 0.38$.

Rule of Mixtures

$$\begin{aligned} E_1 &= E_f\phi_f + E_m(1-\phi_f) \\ &= (76)(0.4) + (3)(1-0.4) = 32.20\text{GPa} \end{aligned}$$

Therefore

$$\epsilon_1 = \frac{\sigma_1}{E_1} = \frac{100\times10^6}{32\times10^9} = 0.00311$$

Poisson's ratio...

$$\nu_{12} = \phi_f\nu_f + \phi_m\nu_m = (0.4)(0.22) - (1-0.4)(0.38) = 0.316$$

So, tensile strain will be:

$$\epsilon_2 = -\nu_{12}\epsilon_1 = -(0.316)(0.00311) = -0.000983 = -9.83\times10^{-4}$$

18 Unidirectional Composite Plate Mechanics

18.1 Sample Calculation: Example 6.3

Apply 15 MPa parallel to axis 2 (normal to fibres) and find the strains.

$$E_2 = \frac{E_fE_m}{E_f(1-\phi_f)+E_m\phi_f} = \frac{(76)(3)}{76(0.6)+3(0.4)} = 4.87\text{GPa}$$

Therefore strain normal to fibres will be

$$\epsilon_2 = \frac{\sigma_2}{E_2} = \frac{15\times10^6}{4.87\times10^9} = 3.08\times10^{-3}$$

The second Poisson's ratio is given by:

$$\nu_{21} = E_2\frac{\nu_{12}}{E_1} = (4.87)\left(\frac{0.316}{32.2}\right) = 0.0478$$

Hence

$$\epsilon_1 = -\nu_{21}\epsilon_2 = -(0.0478)(0.00308) = -1.47\times10^{-4}$$