1 4th Year Materials Engineering

Mechanics of Composite Materials – Lecture 2

2 Last Week

2.1 Summary

Introduced Composites

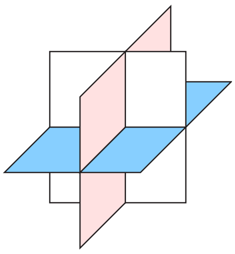
- Structure
  - Bulk matrix
  - Embedded reinforcing materials
- Materials
  - Matrix: Thermoset & Thermoplastic Polymers
  - Reinforcement: Glass, Graphite, Kevlar®
- Manufacturing/Fabrication

Then we introduced the concepts of **isotropy**, **anisotropy**, **symmetry** and **coordinate transformations**.

3 Last Week

In general: Material properties are different depending on direction.
In general such materials are **anisotropic**.
⇔ They have low **symmetry**.

- **Isotropic :** All directions are equivalent.
- **Anisotropic:** i.e. "not isotropic".
  - **Orthotropic:** 3 Orthogonal planes of material symmetry.



  - **Transversely Isotropic:** Axis of rotational symmetry.

4 Symmetry

4.1 Transformations

We will talk about symmetry in terms of transformations. E.g.:

- Mirror reflection in a plane
- Rotation about an axis

Body is symmetric when it is **invariant** under one or more such transformations.

E.g. If something has a plane of mirror symmetry, means that after reflecting it in the plane, it looks the same after as before.

5 Symmetry and Coordinate Transformations

5.1 Mirror Symmetry

$$\text{Transformation Tensor:} \quad \beta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}_{x_1 - x_2 \text{ plane}} \tag{1}$$

Easy to see application to a vector/point $(x, y, z) = (2, 3, 4)$...

$$\boldsymbol{u} = [\beta]\,\boldsymbol{u'} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} 2 \\ 3 \\ 4 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 3 \\ -\ 4 \end{Bmatrix} \tag{2}$$

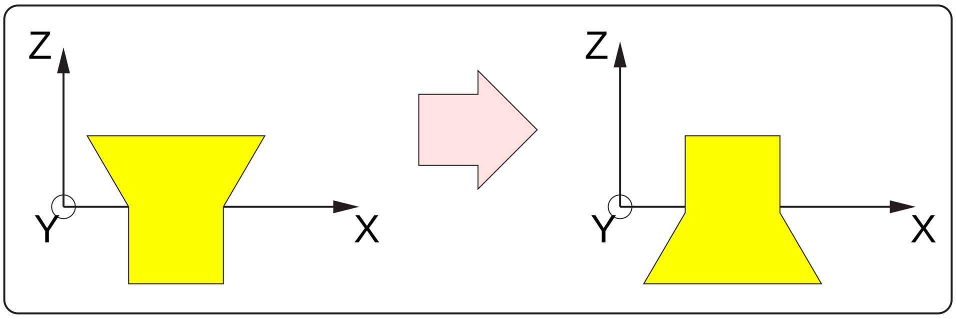
Or using the Einstein Tensor Summation convention

$$u_i = \beta_{ij} u'_j \tag{3}$$

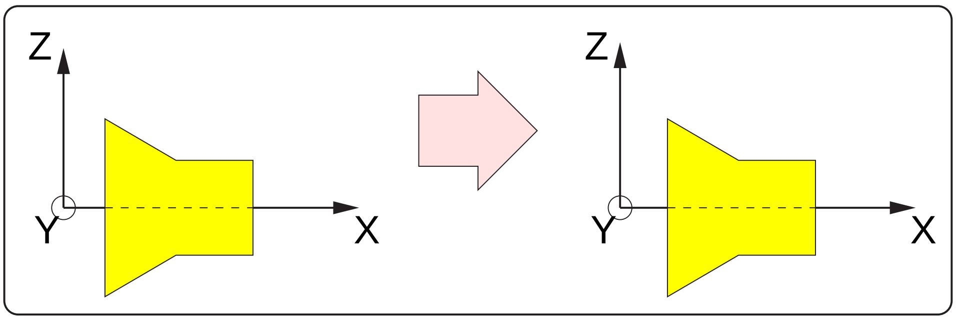
$$\iff u_i = \sum_{j=1}^{3} \beta_{ij} u'_j \qquad \text{for each of } i = 1, 2, 3 \tag{4}$$

6 Symmetry and Coordinate Transformations

6.1 Mirror Reflection, $x - y$ Plane, Asymmetric



6.2 Mirror Reflection, $x - y$ Plane, Symmetric



7 Symmetry and Coordinate Transformations

7.1 Rotation About an Axis

The transformation tensor/matrix is now

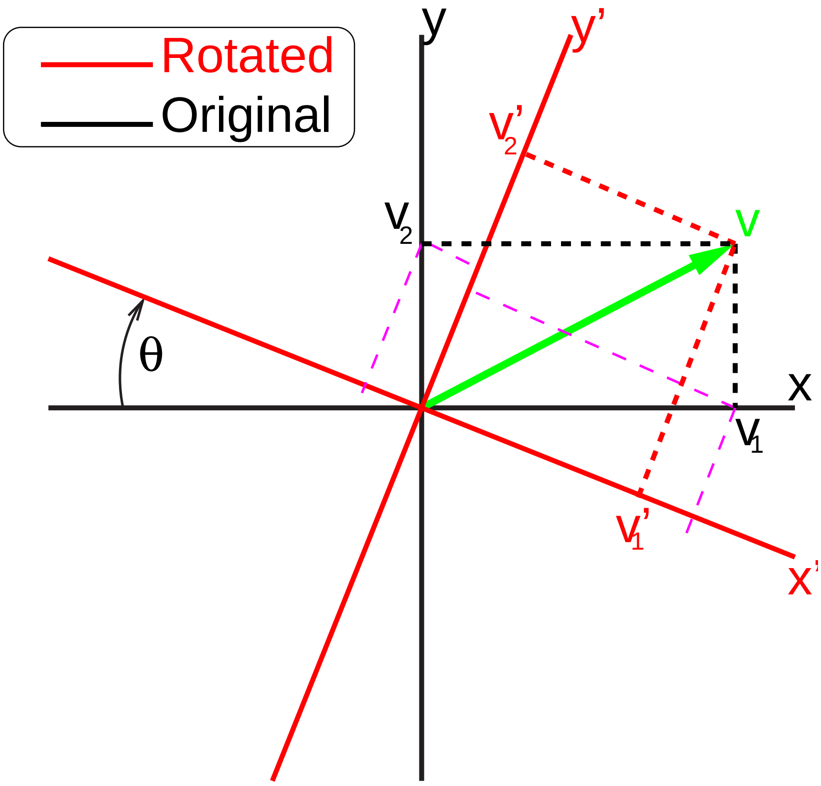
$$\beta_{ij} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5}$$

Angle is θ, but in what direction is the rotation?
If we are rotating the **axes**, then the rotation shown in (**??**) is clockwise.

$$v'_1 = v_1 \cos\theta - v_2 \sin\theta \tag{6}$$
$$v'_2 = v_1 \sin\theta + v_2 \cos\theta \tag{7}$$
$$v'_3 = v_3 \tag{8}$$

Even easier to think in terms of unit vectors. The component v_1 will have two components in the new coordinate system, etc.,

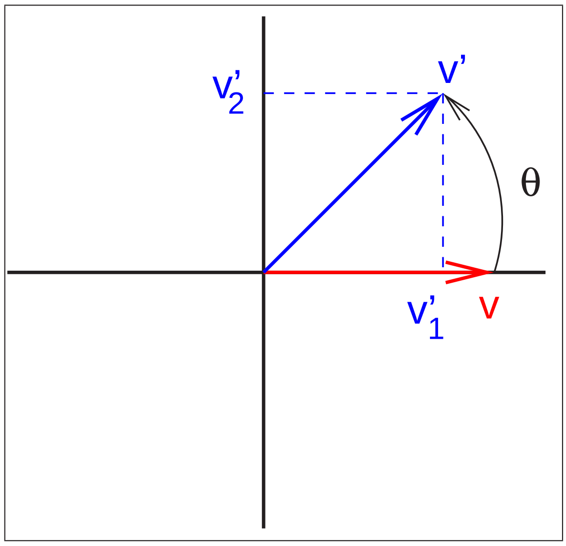
8 Symmetry and Coordinate Transformations



9 Symmetry and Coordinate Transformations

If we instead think about moving the vector $\boldsymbol{v}$ while the coordinate system stays static, then we would say that the vector has been rotated through an angle θ **counter-clockwise**.

$$v'_1 = v\cos\theta$$
$$v'_2 = v\sin\theta$$

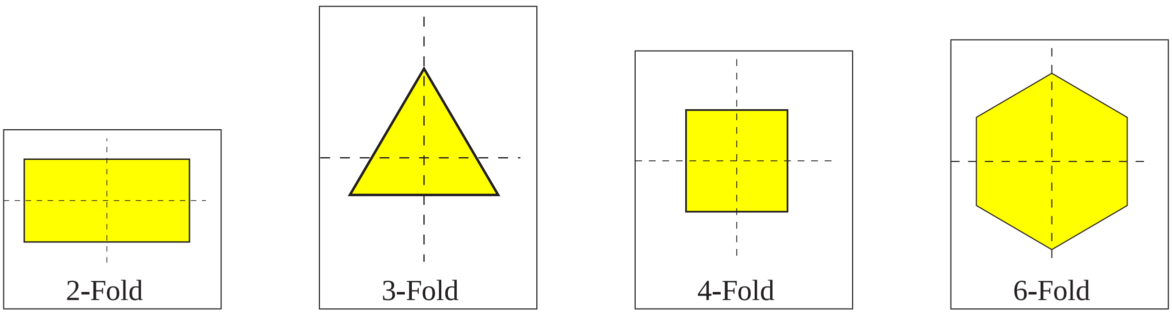


This is perhaps confusing. Fortunately, for definitions of material symmetry, the direction of rotation generally does not matter.

10 Symmetry and Coordinate Transformations

10.1 Rotational Symmetry

Some examples of shapes with rotational symmetries:



In words...

- 2-Fold means rotations of $n \times 360^\circ/2$ (i.e. $180^\circ, 360^\circ, 540^\circ$, etc.,)
- 3-Fold means rotations of $n \times 360^\circ/3$ (i.e. $120^\circ, 240^\circ, 360^\circ$, etc.,)
- 4-Fold means rotations of $n \times 360^\circ/4$ (i.e. $90^\circ, 180^\circ, 270^\circ$, etc.,)
- 6-Fold means rotations of $n \times 360^\circ/6$ (i.e. $60^\circ, 120^\circ, 180^\circ$, etc.,)

11 Symmetry and Coordinate Transformations

11.1 Calculation – Mirror Symmetry

Say we want to find a vector $\boldsymbol{v}$ that is symmetric under a mirror/rotational transform. What does this mean? For mirror symmetry it is simply

$$\boldsymbol{v} = \beta\boldsymbol{v} \iff \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} \tag{9}$$

This gives three equations:

$$v_1 = v_1 \tag{10}$$
$$v_2 = v_2 \tag{11}$$
$$v_3 = -v_3 \tag{12}$$

The first two don't tell us much. But the last one means $v_3 = 0$. v_1 and v_2 can have any values we like.

12 Symmetry and Coordinate Transformations

12.1 Calculation – Rotational Symmetry

We'll do this in the same way as the last time. Assume $\theta = 45°$, so $\sin\theta = \cos\theta = 1/\sqrt{2} \approx 0.7$

$$\boldsymbol{v} = \beta\boldsymbol{v} \iff \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} = \begin{bmatrix} 0.7 & -0.7 & 0 \\ 0.7 & 0.7 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}$$

This (again) gives three equations:

$$v_1 = 0.7v_1 - 0.7v_2 \Leftrightarrow -0.3v_1 - 0.7v_2 = 0$$
$$v_2 = 0.7v_1 + 0.7v_2 \Leftrightarrow 0.7v_1 - 0.3v_2 = 0$$
$$v_3 = v_3$$

The third equation tells us nothing new.
However, the first two tell us that $v_1 = 0$ and $v_2 = 0$.

13 Recap

13.1 Where are we now?

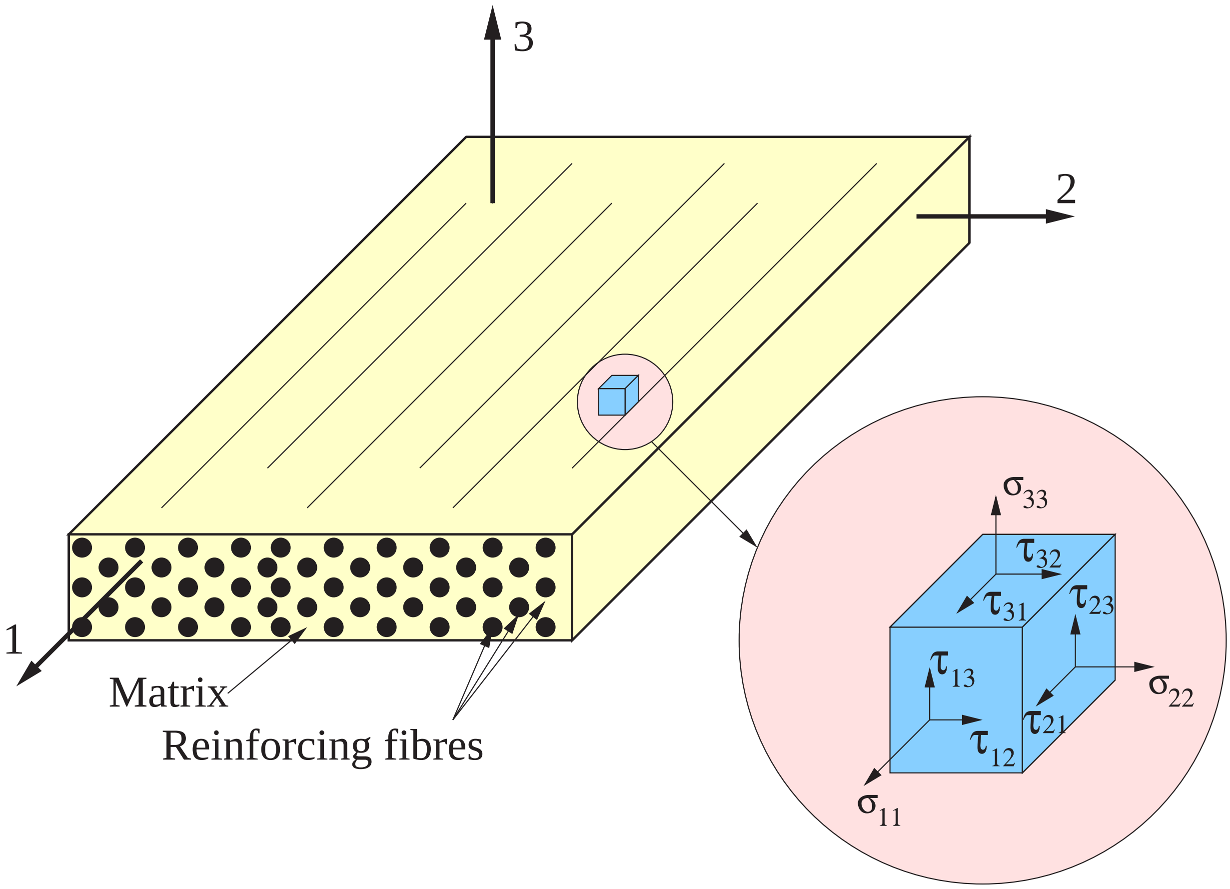
- We have mentioned that materials are often **anisotropic**
- For particular cases of anisotropy we've seen the terms **orthotropic** and **transversely-isotropic**
- Symmetry is related to anisotropy, so we have looked at that in some detail through two transforms:
  - Mirror Reflection in a Plane
  - Rotation about an Axis
- We have seen shapes that are unchanged under these transforms
- We have looked at how to transform a point & what restrictions symmetry requirements can impose.

14 Recap

14.1 Where are we going?

- We will look at material behaviour
  - Stress
  - Strain
  - Stiffness
- We will look at the form these quantities take
- We will look at how symmetry requirements affect these quantitities (in particular stiffness)

15 Stress in 3D



16 Stress

Stress is force per unit area: $\mathrm{N/m^2}$

Stresses (and strains) are given using two subscripts. The first subscript indicates the plane on which the stress acts (it is the plane perpendicular to that axis). The second indicates the direction in which it acts.

- **Normal Stresses:** Perpendicular to the surface on which they act. Two subscripts are the same: σ_{11}, σ_{22} and σ_{33}.
- **Shear Stresses:** Parallel to the surface on which they act. Two subscripts are different, e.g. τ_{12}, τ_{13} and τ_{23}. Note that $\tau_{12} = \tau_{21}$.

17 Strain

17.1 Strain

Strain is a measure of deformation, i.e. how much the shape/geometry of something has changed. It is **dimensionless** i.e. it has no units. You can define strain in terms of change of length of something divided by its original length. Taking a limit gives the strain at a point, this becomes a derivative.

We will use the symbol ϵ for strain. u_i is displacement in direction i, x_i is distance in direction i.

17.2 Normal Strain

Then ϵ_{11}, a normal strain, becomes.

$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1}$$

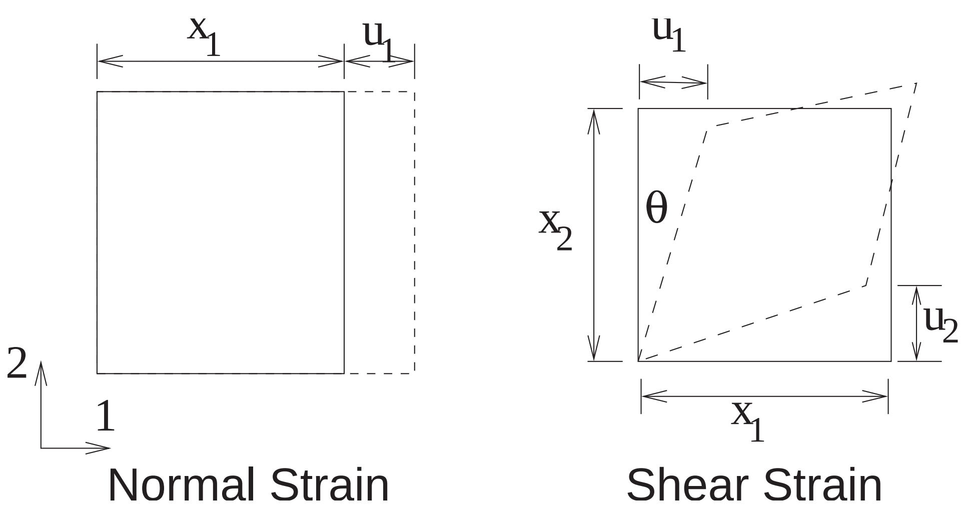
18 Shear Strain

Shear strains are similar, but require more explanation. ϵ_{12} becomes.

$$\epsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$$

This is effectively an angular measure if we assume small deformation. For small deformation u_1:

$$\frac{u_1}{x_2} = \frac{\partial u_1}{\partial x_2} = \tan\theta \approx \theta \text{ (in radians)}$$



19 Stress and Strain Tensors

These two tensors can now be expressed in 3×3 matrix form:

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \quad \epsilon_{ij} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{12} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{13} & \epsilon_{23} & \epsilon_{33} \end{bmatrix} \tag{13}$$

Note the symmetry in these tensors,

$$\sigma_{ij} = \sigma_{ji} \tag{14}$$
$$\epsilon_{ij} = \epsilon_{ji} \tag{15}$$

This will be used later

20 Stiffness

The **stiffness tensor**, and **Hooke's law** tell us how stress and strain are related. i.e. for a given stress, how much deformation will there be. Alternatively, if we measure a given deformation, what is the underlying stress state.
Stiffness tensor:

- in the most general case every strain can contribute to every stress

- tensor has entries of the form c_{ijkl}
- i, j, k, l each take values 1,2,3
- $3 \times 3 \times 3 \times 3 = 81$ entries in total

21 Stiffness Tensor

The meaning of an entry in the tensor is not so confusing though. Take c_{1131} as an example. The significance of this term is that contribution of ϵ_{31} to σ_{11} is given by $(c_{1131})(\epsilon_{31})$

Tensors make this easy to write in a small space as follows:

$$\sigma_{ij} = c_{ijkl}\epsilon_{kl} \tag{16}$$

The trick is that there are a lot of summations going on, so to be clearer:

$$\begin{aligned} \sigma_{ij} = c_{ij11}\epsilon_{11} + c_{ij12}\epsilon_{12} + c_{ij13}\epsilon_{13} + c_{ij21}\epsilon_{21} + c_{ij22}\epsilon_{22} \\ + c_{ij23}\epsilon_{23} + c_{ij31}\epsilon_{31} + c_{ij32}\epsilon_{32} + c_{ij33}\epsilon_{33} \end{aligned} \tag{17}$$

This **summation over repeated indices** (in this case k and l) is used widely in tensor notation. It can initially be confusing, but makes the notation very compact. Also, it is very easy to program a computer to perform these calculations (nested loops).

22 Simplifications

The preceding slides' material can be simplified. Recall from a little earlier that the stress and strain tensors are symmetric. Then, looking at the stiffness tensor in its **most general** form, for any given c_{ijkl} the following is true due to symmetry in the stress and strain tensors:

$$c_{ijkl} = c_{jikl} = c_{ijlk} = c_{jilk} \tag{18}$$

Energy considerations lead to the following additional condition

$$c_{ijkl} = c_{klij} \tag{19}$$

These two requirements mean that there are **at most 21 independent stiffness constants**. The long sum from the previous slide for σ_{ij} then becomes:

$$\begin{aligned} \sigma_{ij} = c_{ij11}\epsilon_{11} + c_{ij22}\epsilon_{22} + c_{ij33}\epsilon_{33} + 2c_{ij12}\epsilon_{12} + 2c_{ij13}\epsilon_{13} + 2c_{ij32}\epsilon_{32} \\ \Leftrightarrow \sigma_{ij} = c_{ij11}\epsilon_{11} + c_{ij22}\epsilon_{22} + c_{ij33}\epsilon_{33} + c_{ij12}\gamma_{12} + c_{ij13}\gamma_{13} + c_{ij32}\gamma_{32} \end{aligned} \tag{20}$$

γ is called the **engineering shear strain** and is simply defined as

$$\gamma_{ij} = 2\epsilon_{ij}, \text{ where } i \neq j. \tag{21}$$