Impulse and Momentum

Linear Momentum

for a particle $\vec{G} = m\vec{v}$ vector quantity

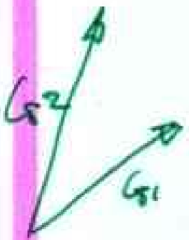
for a system of particles

$$\vec{G} = \sum m_i \vec{v}_i$$

note $\vec{v}_i = \frac{d\vec{r}_i}{dt}$ $\vec{r}_i$ is pos'n vector of m_i one particle in sys.

$$\therefore \vec{G} = \sum m_i \dot{\vec{r}}_i = \frac{d(\sum m_i r_i)}{dt}$$ for constant mass

$$\Rightarrow \vec{G} = m\vec{v}_G$$ $\vec{v}_G$ is velocity of centre of MASS.



Newton's 2nd Law $\vec{F} = m\vec{a}$ can be written

Resultant force $\sum \vec{F} = \dot{\vec{G}}$ so $\int_{t_1}^{t_2} \sum \vec{F}\, dt = \vec{G}_2 - \vec{G}_1$ vector

we can write scalar equations for components

e.g. $\sum F_x = \dot{G}_x$ & $\int_{t_1}^{t_2} \sum F_x = G_{x_2} - G_{x_1}$ etc for y, z.

we call $\int_{t_1}^{t_2} \sum \vec{F}\, dt$ the linear impulse on the body during the time-interval t_1 to t_2

Linear impulse = change in momentum for a given time interval

Angular Momentum

defined as the moment of linear momentum

e.g. $\vec{H}_G = \sum \vec{\rho}_i \times m_i \vec{v}_i$

$\vec{\rho}_i$ vector from centre of mass to m_i

$\vec{v}_i$ velocity of m_i

m_i mass of particle

for 2-D Problems + RIGID BODIES...

$$\boxed{H_G = I_G \omega}$$

← scalar eqn for 2-D.

also we have

$$\boxed{\begin{aligned} \sum M_G &= \dot{H}_G \\ \int_{t_1}^{t_2} \sum M_G \, dt &= H_{G_2} - H_{G_1} \end{aligned}}$$

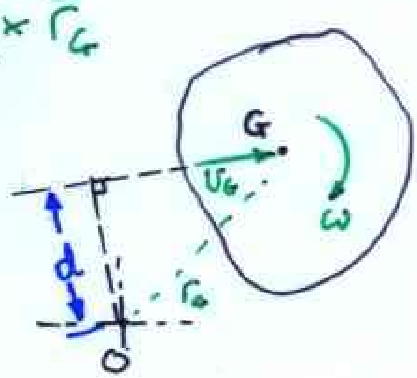
← {angular impulse}

Note H_G is angular momentum about an axis through the centre of mass

FOR AN ARBITRARY POINT "O"

$$\boxed{H_O = I_G \omega + m v_G d}$$

$\vec{V}_G \times \vec{r}_G$



I_G : moment of inertia about G

$\bar{v}_G$: velocity of centre of mass

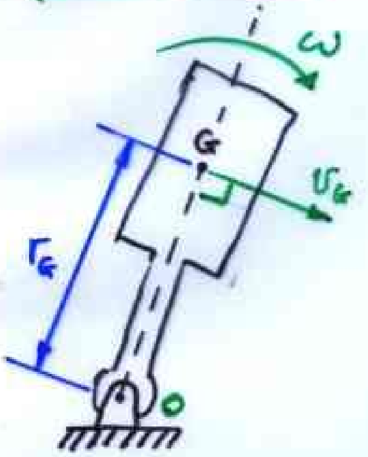
d : perpendicular distance from O to line of action of $\vec{v}_G$

m : mass of BODY.

IF O is a fixed point

$$\boxed{\begin{aligned} H_0 &= I_0 \omega \\ \sum M_0 &= I_0 \dot{\omega} \equiv I_0 \alpha \\ \int_{t_1}^{t_2} \sum M_0 \, dt &= I_0(\omega_2 - \omega_1) \end{aligned}}$$

I_0 moment of inertia about point o



note $\vec{v}_G$ must be perpendicular to $\vec{r}_G$ due to kinematics

CONSERVATION OF MOMENTUM

IF FOR A BODY OR SYSTEM OF BODIES, NO EXTERNAL NET FORCES ARE ACTING

i.e. $\sum \vec{F} = 0$

then $\Delta \vec{G} = 0$ i.e. momentum does not change

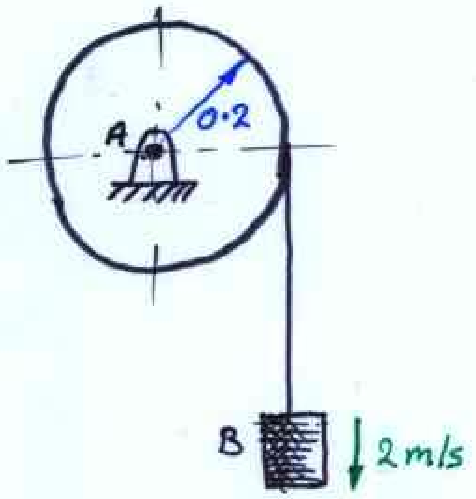
Also, if resultant moment about "O" or about G is zero i.e. $\sum M_0 = 0$ OR $\sum M_G = 0$

$\Rightarrow$

$\Delta H_0 = 0$ or $\Delta H_G = 0$

EXAMPLE

Disk: 20 kg
$r = 0.2\,m$
$I_A = 0.40\ kg\,m^2$ about axis at A



Block: 6 kg
$v_B = 2\ m/s$ down

cord is wrapped around disk

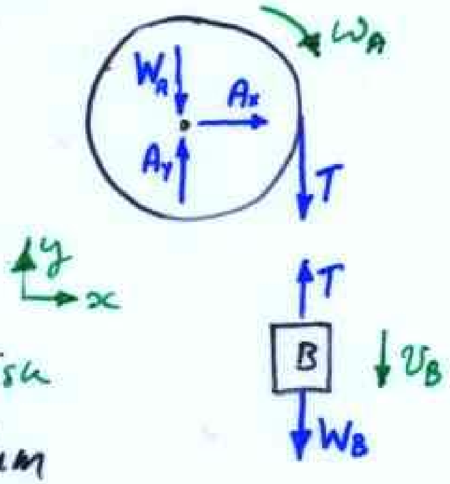
find v_B 3s later

SOLUTION I

Begin by drawing Free Body Diagram

$W_A = (20)(9.81) = 196.2\,N$

$W_B = (6)(9.81) = 58.86\,N$



We can eliminate A_x, A_y, W_A for Disk by applying angular impulse-momentum Equation about point A.

"1" @ start of time
"2" after 3s

Disk

$$I_A\omega_1 + \sum\int_1^2 M_A\,dt = I_A\omega_2$$

$$(0.40)\omega_1 + (T)(3)(0.2) = 0.40\,\omega_2 \quad (A)$$

for Block..

$$m_B v_{B1} + \int_1^2 \sum F_y\,dt = m_B v_{B2}$$

$$-6(2) + (T)\overset{\Delta t}{(3)} - (58.86)\overset{\Delta t}{(3)} = -6(v_{B2}) \quad (B)$$

Kinematics :

$v_B = r\omega$ because there is no slip

$\Rightarrow \omega_1 = \frac{2}{0.2} = 10 \text{ rad/s}$

$\Rightarrow \omega_2 = 5\, v_{B2}$

Substitute in ...

$(0.40)(10) + (0.6)T = (2.0)v_{B2}$ (A)

$-(6)(2) + 3T - 176.58 = -6\, v_{B2}$

Tidying up $\Rightarrow$ 2 Simultaneous Eqns.

$4 + 0.6T = 2.0\, v_{B2}$

$-182.78 + 3T = -6\, v_{B2}$

Solve for v_{B2}

$\Rightarrow v_{B2} \approx 13.0 \text{ m/s}$

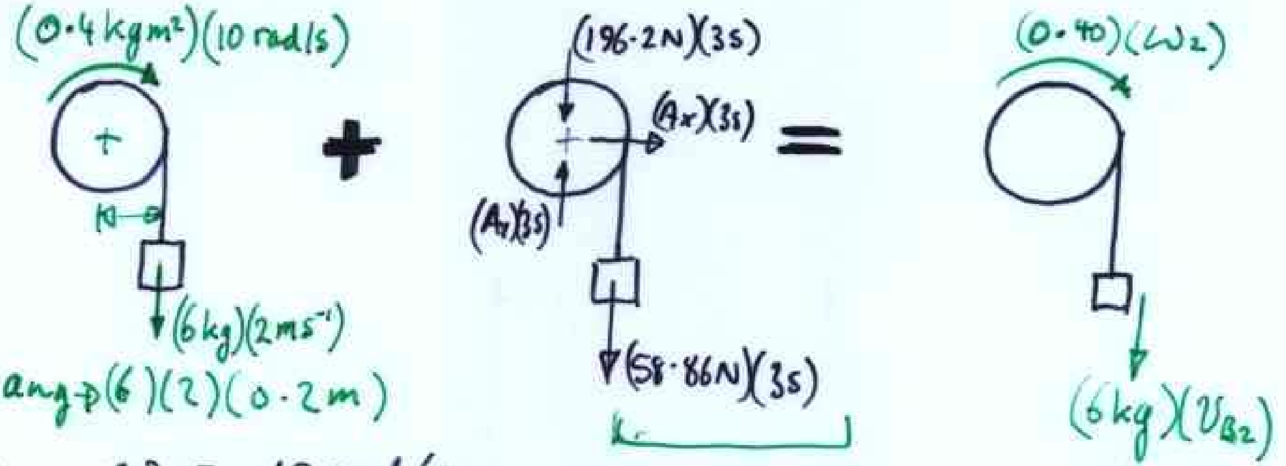
Solution II

We can also consider block, cord

and disk all at once ...

IN this way

$$\left(\sum \begin{matrix}\text{SYS ANG.}\\ \text{MOMENTUM}\end{matrix}\right)_{A,1} + \left(\sum \begin{matrix}\text{SYS. ANG}\\ \text{IMPULSE}\end{matrix}\right)_{A\,1\to 2} = \left(\sum \begin{matrix}\text{SYS ANG}\\ \text{MOMENTUM}\end{matrix}\right)_{A,2}$$



Again $\omega_1 = 10 \text{ rad/s}$

$\omega_2 = 5 v_{B2}$

$$\overset{\text{BLOCK}}{} \qquad \overset{\text{DISC}}{}$$

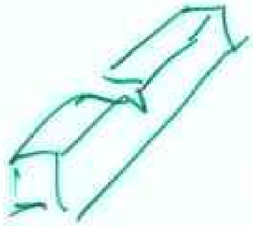
$$(+)\ \left[(6\,kg)(2\,m/s)(0.2\,m) + (0.40\,kgm^2)(10\,rad/s)\right]$$

$$+\left[(58.86\,N)(3\,s)(0.2\,m)\right]$$

$$= \left[(6\,kg)(v_{B2})(0.2\,m) + (0.40\,kgm^2)(5 v_{B2})\right]$$

Solve to get

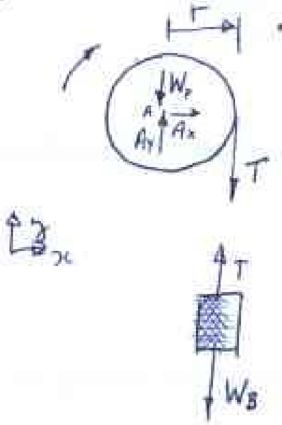
$$\underline{v_{B2} \approx 13.0 \text{ m/s}}$$



Alternative approach to pulley & block problem

in class, an impulse & momentum approach was followed but there are other ways to solve problem

Again, DRAW F.B.D. for Block & for pulley



W_p is weight of pulley

T is tension in cord

W_B is weight of block

for pulley $\Sigma M_A = I_A \alpha$

$\Rightarrow$ (+) $Tr = I_A \alpha$

& $T = \dfrac{I_A \alpha}{r}$ (*)

for block $\Sigma F_y = m a_y$ $\downarrow +$ $\Rightarrow W_B - T = m a_y$

note $W_B = mg$ $\Rightarrow T = W_B - m a_y$ (Q)

note, from kinematics that a_y for block is related to α

$a_y = r\alpha$ & $\alpha = \dfrac{a_y}{r}$ subst into (*)

$\Rightarrow$ $T = \dfrac{I_A a_y}{r^2}$ (P)

equate (P) & (Q) $\dfrac{I_A a_y}{r^2} = W_B - m a_y$

$\Rightarrow a_y = \dfrac{W_B}{\left(\frac{I_A}{r^2} + m\right)} = \dfrac{(6)(9.81)}{\frac{0.4}{(0.2)^2} + 6} = 3.68\ m s^{-2}$

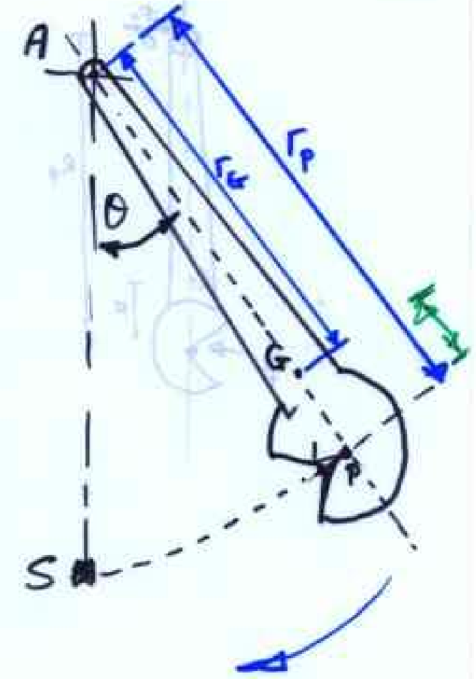
$V_{B_2} = V_{B_1} + at$

$= 2 + (3.68)(3)$

note that this is effectively an integration

$\boxed{V_{B_2} = 13.04\ m/s}$ same answer

CHARPY IMPACT TESTER

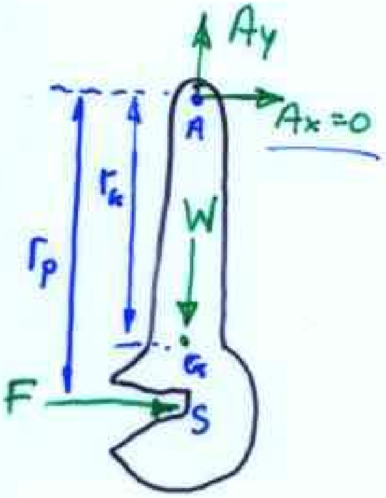


PENDULUM SWINGS & STRIKES TARGET AT S.

find r_p to minimise Horiz force at A during impact

Simplify calc. by assuming specimen stops the pendulum.

FREE BODY DIAGRAM
at moment of impact



ang vel just before impact is ω_1 c.w.

v_{G_1} is $r_G \omega_1$, to left ←

after impact $v_{G_2} = 0$ & $\omega_2 = 0$

⇒ angular momentum EQN

$$\boxed{\circlearrowright +} \quad I_A \omega_1 - \left(\int F\,dt\right) r_p = 0 \qquad \cancel{I_A \omega_2}^{\,0}$$

linear momentum

$$\xrightarrow{+} \quad -m(\overbrace{r_G \omega_1}^{v_{G_1}}) + \left(\int F\,dt\right) = 0 \qquad [\text{note } A_x = 0]$$

SOLVE BETWEEN 2 EQNS, Eliminating $\int F\,dt$:

$$I_A \cancel{\omega_1} - m r_G \cancel{\omega_1} r_p = 0$$

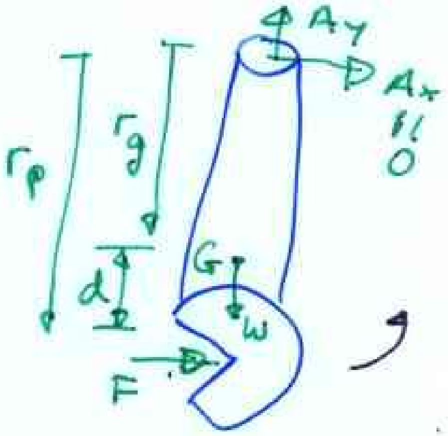
note $I_A = m k_G^2 + m r_G^2$
radius of gyration
$I_A = m k_A^2$

$$\Rightarrow r_p = r_G + \frac{k_G^2}{r_G}$$

$$r_p = \frac{r_G^2 + k_G^2}{r_G} = \frac{k_A^2}{r_G}$$ } **centre of percussion**

ALTERNATIVE APPROACH

$d = r_P - r_g$



$\sum M_G = I_G \alpha$

$\sum M_G = m k_g^2 \alpha$

$\left[F d + \cancel{A_x r_g}^{0} = m k_g^2 \alpha \right]$

$\sum F_x = m a_{Gx}$ Kinematics $a_{Gx} = r_g \alpha$

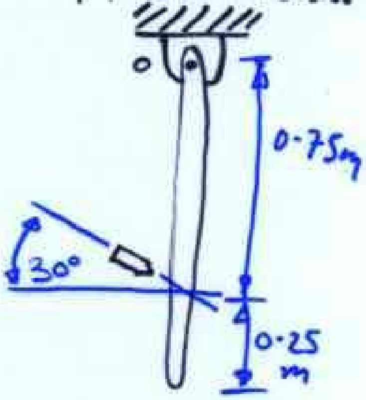
$F = m a_{Gx}$

$\left[F = m r_G \alpha \right]$

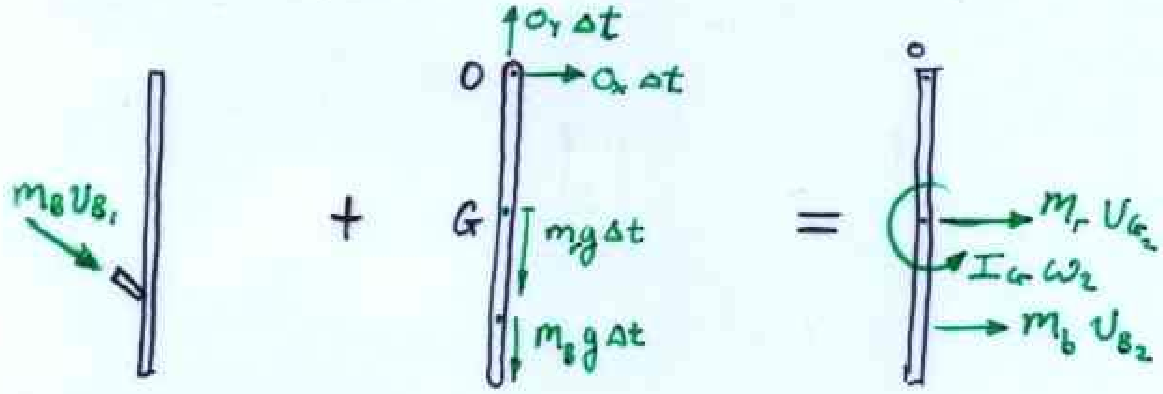
$\cancel{m} r_g \cancel{\alpha} d = \cancel{m} k_g^2 \cancel{\alpha}$

$d = \dfrac{k_g^2}{r_g}$ $r_p = r_g + d = r_g + \dfrac{k_g^2}{r_g}$

BULLET, $v_B = 400$ m/s embeds in 5kg SLENDER ROD AS SHOWN find ω of ROD just after BULLET EMBEDS. (mass of Bullet = 4g)



ANALYSE BULLET & ROD AS A SINGLE SYSTEM



From this we see that there is no angular impulse about O ~~about G~~.

$$\Sigma (H_o)_1 = \Sigma (H_o)_2$$

$$\circlearrowleft + \quad m_B v_{B_1} \cos(30°)(0.75) = m_B v_{B_2}(0.75m) + \underbrace{\left[m_R v_{G_2}(0.5m) + I_G \omega_2 \right]}_{I_O}$$

but note
$$\left. \begin{aligned} v_{B_2} &= 0.75\,\omega_2 \\ v_{G_2} &= 0.5\,\omega_2 \\ I_G &= \tfrac{1}{12} m l^2 \end{aligned} \right] \text{Kinematics}$$

solve & get $\omega_2 = 0.62$ rad/s

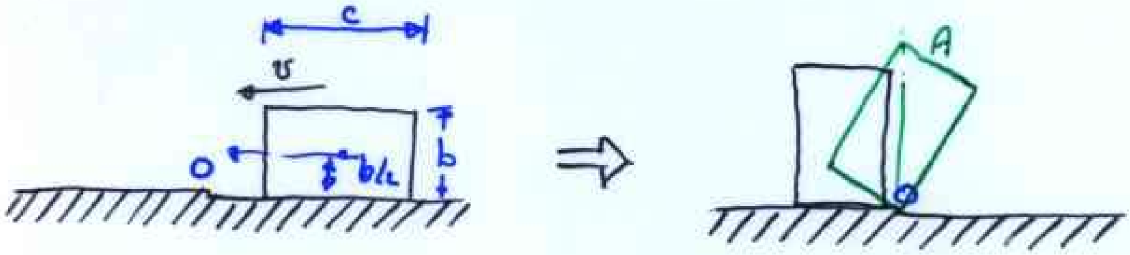
Approximation useful...

neglect bullet on R.H.S.

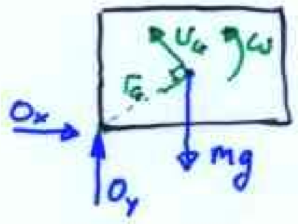
$$(m_B)(v_{B_1})(\cos 30°)(0.75) \approx I_O \omega_2 = \tfrac{1}{3} m l^2 \omega^2$$

$$\Rightarrow \omega_2 \approx \frac{(0.004)(400)(0.866)(0.75)}{\tfrac{1}{3}(5)(1.0)^2} = 0.624$$

UNIFORM BLOCK SLIDING to LEFT strikes a small step as SHOWN. FIND minimum v s.t. BLOCK CAN PIVOT AND reach STANDING POSN A SHOWN w/ no velocity & % energy loss if $b=c$



FREE BODY DIAGRAM



INITIAL ANGULAR MOMENTUM...
just before impact

$$H_{o_1} = mv\left(\frac{b}{2}\right)$$

JUST AFTER IMPACT...

vel of c.o.g. is v_G as SHOWN on F.B.D.

ang vel is $\omega = v_G / r_G$ $\qquad H_o = I_o\omega$

$$\Rightarrow H_{o_2} = \Big(\underbrace{\tfrac{1}{12}m(b^2+c^2)}_{I_G} + m\underbrace{\left[\left(\tfrac{c}{2}\right)^2+\left(\tfrac{b}{2}\right)^2\right]}_{(r_G)^2}\Big)\omega$$

I_o

$$= \frac{m}{3}(b^2+c^2)\omega = I_o\omega$$

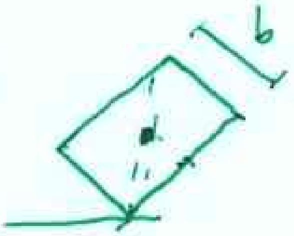
NEGLECT ANGULAR IMPULSE OF WEIGHT (TIME of collision is short so $\int_1^2 mg\,dt$ will be small).

$\Rightarrow$ Apply CONSERVATION of MOMENTUM... $\Delta H_o = 0$

$$\Rightarrow \frac{m}{3}(b^2+c^2)\omega = mv\left(\frac{b}{2}\right) \Rightarrow \left[\omega = \frac{3vb}{2(b^2+c^2)}\right] *$$

ONCE BLOCK STARTS PIVOTING, ENERGY WILL BE CONSERVED

$$\Rightarrow \underset{\text{KINETIC}}{\Delta T} + \underset{\text{POTENTIAL}}{\Delta V_g} = 0$$



in POSN A

$$\frac{1}{2} I_o \omega^2 - mg\left[\sqrt{\left(\frac{b}{2}\right)^2 + \left(\frac{c}{2}\right)^2} - \frac{b}{2}\right] = 0$$

$$\frac{1}{2}\,\frac{m}{3}(b^2+c^2)\left[\frac{3vb}{2(b^2+c^2)}\right]^2 - \frac{mg}{2}\left[\sqrt{b^2+c^2} - b\right] = 0$$

$$\Rightarrow v = 2\sqrt{\frac{g}{3}\left(1+\frac{c^2}{b^2}\right)\left(\sqrt{b^2+c^2} - b\right)}$$

% ENERGY LOSS:

radius of gyration

$$\frac{\Delta E}{E} = \frac{\frac{1}{2} m v^2 - \frac{1}{2} I_o \omega^2}{\frac{1}{2} m v^2} = 1 - \frac{k_o^2 \omega^2}{v^2}$$

$$= 1 - \left(\frac{b^2+c^2}{3}\right)\left[\frac{3b}{2(b^2+c^2)}\right]^2$$

SUBST from (*)

$$= 1 - \frac{3}{4\left(1+\frac{c^2}{b^2}\right)}$$

if $b=c$

$$\Rightarrow \frac{\Delta E}{E} = 62.5\%$$