PLANE Kinetics of Rigid Bodies

WORK and ENERGY

work done by a force $\vec{F}$ is

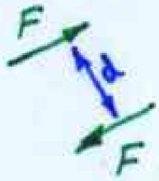
$$\boxed{U = \int \vec{F} \cdot d\vec{r}} \quad \text{OR} \quad \boxed{U = \int F \cos(\phi)\, ds} \qquad \text{SCALAR}$$

SCALAR-DOT PRODUCT $\qquad$ ϕ is angle between $\vec{F}$ & $d\vec{r}$

WORK DONE BY a COUPLE M is

$$\boxed{U = \int M \, d\theta}$$

RECALL, A COUPLE is 2 opposite forces $\qquad M = Fd$



if we TRANSLATE A couple, work DONE BY Forces CANCELS out (or is zero)

Kinetic Energy:

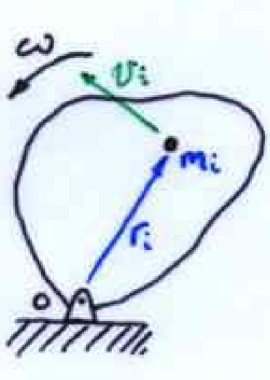
TRANSLATION: $\boxed{T = \frac{1}{2} m v^2}$

Fixed Axis Rotation:

CONSIDER PARTICLE m_i

$v_i = r_i \omega$

K.E. of Particle is $T_i = \frac{1}{2} m_i r_i^2 \omega^2$



for WHOLE BODY

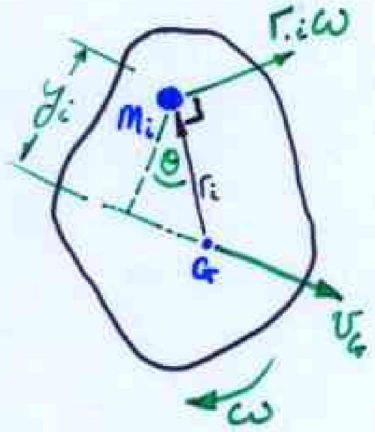
$$T = \sum T_i = \sum \tfrac{1}{2} m_i r_i^2 \omega^2 = \tfrac{1}{2}\omega^2 \sum m_i r_i^2$$

$m_i \to dm \Rightarrow$ integration

$$T = \tfrac{1}{2}\omega^2 \underbrace{\int m_i r_i^2}_{I_o \;\to\; \int r_i^2 dm} \Rightarrow \boxed{T = \tfrac{1}{2} I_o \omega^2}$$

Kinetic Energy

General Plane Motion



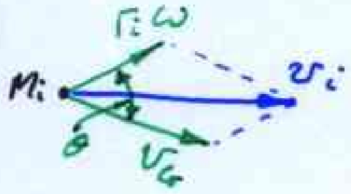
velocity of mass centre is v_G

angular velocity is ω

Particle m_i has 2 components of vel

$$\vec{v}_i = \vec{v}_G + \vec{r}_i \times \vec{\omega}$$

vector addition



$$v_i = \sqrt{v_G^2 + (r_i\omega)^2 + 2 v_G r_i \omega \cos\theta}$$

Cosine Rule

θ is angle between $r_i\omega$ & v_G

$$\therefore T_i = \tfrac{1}{2} m_i v_i^2$$

from c.o.g

$$= \tfrac{1}{2} m_i \left[v_G^2 + r_i^2\omega^2 + \underline{2 v_G r_i \omega \cos\theta} \right]$$

Sum / integrate ; 3rd term → zero by Defn of C of G.

$$\Rightarrow T = \tfrac{1}{2} m v_G^2 + \tfrac{1}{2}\omega^2 \int m_i r_i^2 \quad \int r_i^2 \, dm$$

$$\Rightarrow \boxed{T = \tfrac{1}{2} m v_G^2 + \tfrac{1}{2} I_G \omega^2}$$

if you know a point that has instantaneous ~~instant~~ zero velocity, then you can also say

$$\boxed{T = \tfrac{1}{2} I_c \omega^2}$$

I_c moment of inertia about that point

Potential Energy:

gravitational : $\boxed{V_g = mgh \ \& \ \Delta V_g = mg\,\Delta h}$

elastic : $\boxed{\begin{array}{l} V_e = \int_0^x kx\,dx = \frac{1}{2}kx^2 \\ \Delta V_e = \frac{1}{2}k\left(x_2^2 - x_1^2\right) \end{array}}$

k = spring const.
x = extension of spring

Work energy equation:

$$\boxed{U'_{1\to2} = \Delta T + \Delta V_g + \Delta V_e}$$

U' denotes work done by all forces EXCEPT weight (gravity) and elastic forces.

these are accounted for in ΔV_e & ΔV_g

Power

work done per unit time

force $\quad P = \dfrac{dU}{dt} = \dfrac{\vec{F}\cdot d\vec{r}}{dt} = \underbrace{\vec{F}\cdot\dfrac{d\vec{r}}{dt}}$ $= \vec{F}\cdot\vec{v}$ ← velocity of point of application of $\vec{F}$

– at any instant.

assumed that $\vec{F}$ is not varying w.r.t time

Couple $\quad P = \dfrac{dU}{dt} = M\dfrac{d\theta}{dt} = M\omega$

Total $\boxed{P = \underbrace{M\omega + \vec{F}\cdot\vec{v}}}$ — SCALAR QUANTITY

↓ SCALAR calculation Because we're working in 2D

Power can also be expressed as rate at which total mechanical energy of a rigid BODY or SYSTEM of such BODIES IS CHANGING

$$P = \frac{dU'}{dt} = \dot{T} + \dot{V}_g + \dot{V}_e = \frac{d}{dt}(T+V)$$

i.e.

Power Developed By active Forces & Couples = Rate of change of Mechanical Energy.

Note:

$$\dot{T} = \frac{dT}{dt} = \frac{d}{dt}\left(\frac{1}{2}m\vec{v}\cdot\vec{v} + \frac{1}{2}I_G\omega^2\right)$$

$$= \frac{1}{2}m\left(\vec{a}\cdot\vec{v} + \vec{v}\cdot\vec{a}\right) + I_G\omega\dot{\omega}$$

$$= m\vec{a}_G\cdot\vec{v}_G + I_G\alpha\omega$$

$$\dot{T} = \vec{R}\cdot\vec{v}_G + M_G\omega$$

$\vec{R}$ is resultant force of ALL forces acting on BODY

$\vec{M}$ is resultant moment about G of ALL forces.

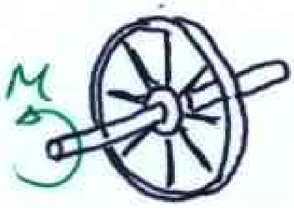
EXAMPLE: 6/124

50 kg FLYWHEEL; radius of gyration = 0.4 m about shaft

TORQUE APPLIED: $M = 2(1 - e^{-0.1\theta})$ N·m θ in radians

@ $\theta = 0$, flywheel @ rest

after 5 revs what's ω?



ENERGY EQUATION:

$T_2 = T_1 + U_{1\to2}$

FINAL kin. Energy

kinetic energy @ START

WORK DONE

NO POTENTIAL ENERGY

$T_1 = 0$ because $\omega = 0$ @ $\theta = 0$

find $U_{1\to2}$

$$U_{1\to2} = \int_1^2 M\,d\theta = \int_0^{10\pi} 2(1 - e^{-0.1\theta})\,d\theta$$

$$= \left[2\theta + 20e^{-0.1\theta}\right]_0^{10\pi}$$

$$= \left((2)(10\pi) + 20e^{-0.1(10\pi)}\right) - (20)$$

$U_{1\to2} = \underline{43.7}$ J $= T_2$

$I_G = mk_G^2$

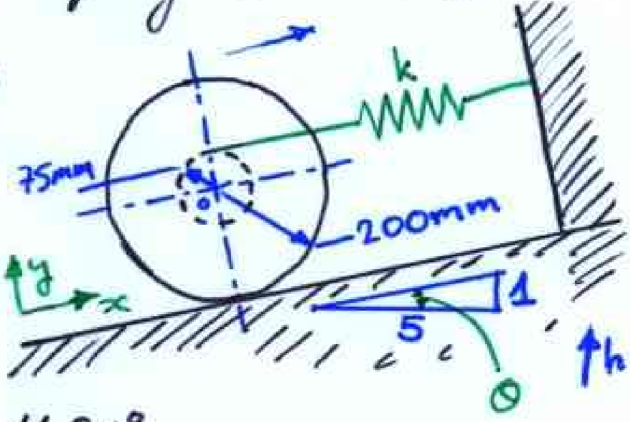
$$T_2 = \frac{1}{2}I_G\omega^2 = \frac{1}{2}\underbrace{(50)}_{m}\underbrace{(0.4)^2}_{k_g^2}\omega^2 = 4\omega^2$$

$\therefore\ 43.7 = 4\omega^2 \Rightarrow \boxed{\omega = 3.31 \text{ rad/s}}$

Example 6/142

10kg double wheel, radius of gyration $k_o = 125\text{ mm}$

Spring stiffness $k = 600\text{ N/m}$. Spring connects to wheel by cord ~~WR~~ WRAPPED AROUND inner hub. When spring is stretched 225mm, wheel is released. find max vel of O



WHAT is θ? $\quad \theta = \tan^{-1}\left(\frac{1}{5}\right) = 11.31°$

$$\Rightarrow \sin\theta = 0.1961 = \frac{1}{\sqrt{1^2+5^2}} \text{ (alternative).}$$

if wheel moves... how is potential Energy changed

gravity: $\Delta V_g = mg\Delta h = mgx\sin\theta$

spring: $\Delta V_e = \frac{1}{2}k(l_2^2 - l_1^2)$ $\quad l$ being spring extension.

Need to be careful. WHEEL moves Distance x in positive x Direction ⇒ SPRING RELAXES HOW MUCH?

2 COMPONENTS: translation ... x

unwinding ... $r\theta$

look at $r\theta$ ... $r = 0.075$ m

$\theta = \frac{x}{0.200}$ ← outer radius

∴ total relaxation is

$$x + 0.075\frac{x}{0.200}$$

$$= \left(\frac{0.275}{0.200}\right)x$$

returning to

$$\Delta V_e = \tfrac{1}{2}k\left(l_2^2 - l_1^2\right)$$

we know $l_1 = 0.225\,m$

$$l_2 = l_1 - \text{relaxation} = 0.225 - \frac{275}{200}x$$

$$\therefore\ \Delta V_e = \tfrac{1}{2}k\left[\left(0.225 - \frac{275}{200}x\right)^2 - \left(0.225\right)^2\right]$$

WE NOW HAVE ΔV_e & ΔV_G IN TERMS of x

NOW LOOK @ KINETIC ENERGY:

$$\Delta T = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I_G\omega^2 \qquad \text{NOTE } \omega = \left(\frac{v}{r_o}\right)$$

Rel to zero vel.

outer radius of wheel

$$= \tfrac{1}{2}(10)v^2 + \tfrac{1}{2}(10)(0.125)^2\left(\frac{v}{0.2}\right)^2$$

$$\underline{\Delta T = 6.95\,v^2}$$

OVERALL NO WORK (other than grav & elasticity)

$$\Rightarrow\ \underline{\Delta T + \Delta V_G + \Delta V_e = 0}\ \ * \qquad U'_{1\text{-}2} = 0 \ \boxed{\text{in this case!}}$$

$$\Delta V_G = mgx\sin\theta = (10)(9.81)(0.1961)x = 19.24x$$

$$\Delta V_e = \tfrac{1}{2}(600)\left[\left(0.225 - \frac{275}{200}x\right)^2 - (0.225)^2\right] = 567.2x^2 - 185.6x$$

* then gives

$$6.95v^2 + 19.24x + 567.2x^2 - 185.6x = 0$$

$$\Rightarrow\ v^2 = 23.93x - 81.57x^2 \qquad v\ \text{max @ } v^2\ \text{max}$$

$$\frac{dv^2}{dx} = 0 = 23.93 - (2)(81.57)x$$

$$\Rightarrow\ x = \underline{0.1467}\ m \quad \text{for } v_{max}$$

$$v_{max}^2 = (23.93)(0.1467) - (81.57)(0.1467)^2$$

$$= 1.755$$

$$\left[\ V_{max} = \sqrt{1.755} = 1.325\ m/s\ \right]$$

EXAMPLE Q. 6/150

(G8)

CONSTRAINED MOTION

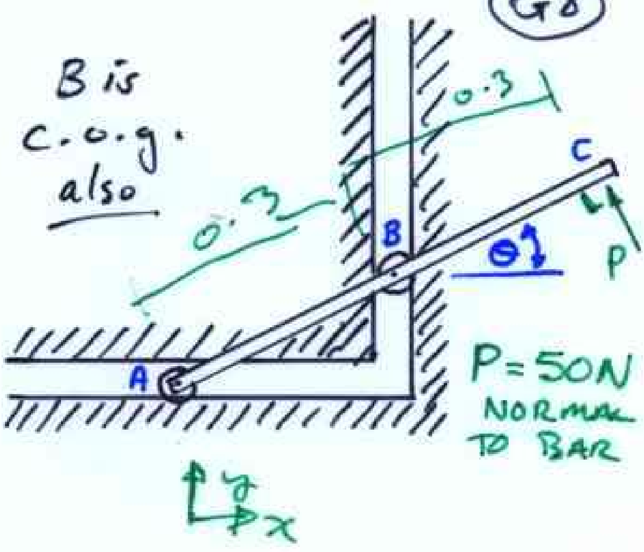
BAR LENGTH = 600 mm

MASS = 4kg

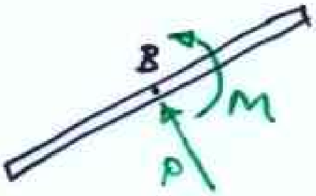
starts @ $\theta = 0$

find velocity v at which A will strike vertical support. at $\theta = 90°$.

B is c.o.g. also



Replace force P by a force + couple @ B



$M = (50)(0.3) = 15\,Nm$

$U'_{1-2} = \Delta T + \Delta V_g$ no Ve term

ΔV_g is easy ... $\Delta V_g = mg\Delta h = (4)(9.81)(0.3) = 11.77\,J$

ΔT NEEDS A LITTLE MORE THOUGHT.

when $\theta = 90°$ center of mass is STATIC

OBVIOUSLY $v_{Gx} = 0$ ... kinematics

$v_{Gy} = 0$ because it is @ peak of curve

$\therefore \Delta T = \frac{1}{2} I_B \omega^2 = \frac{1}{2}\left(\frac{1}{12}(4)(0.6)^2\right)\omega^2$ SUBSTITUTE IN.

v_A is vel of A $\Rightarrow \omega = \frac{v_A}{0.3}$

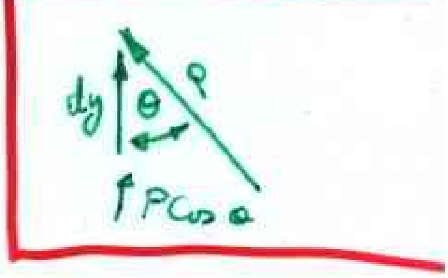
$v = r\omega$

$\Delta T = 0.667\, v_A^2$

$U_{1\to 2} = \int_1^2 \vec{F}\cdot d\vec{r} + \int M\, d\theta$

$= \int_0^{0.3} 50 \cos\theta\, dy + \int_0^{\frac{\pi}{2}} 15\, d\theta$

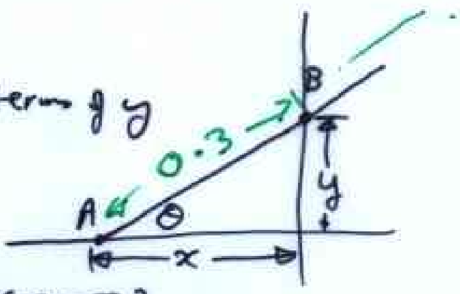
M



$$\int_0^{0.3} 50 \cos\theta \, dy$$

we could EXPRESS $\cos(\theta)$ in term of y

$$\cos(\theta) = \frac{\sqrt{(0.3)^2 - y^2}}{0.3} = \frac{x}{0.3}$$



HOW WOULD SUCH A TERM BE INTEGRATED?

Better to put dy in terms of θ

$$y = 0.3 \sin(\theta)$$

$$\frac{dy}{d\theta} = 0.3 \cos\theta \Rightarrow dy = 0.3 \cos\theta \, d\theta$$

what will Limits be? $y = 0 \Rightarrow \theta = 0$, $y = 0.3 \Rightarrow \theta = \frac{\pi}{2}$ CAN SEE FROM GEOMETRY OR SUBSTITUTION

So we have

$$U'_{1\to2} = \int_0^{\frac{\pi}{2}} 15 \cos^2\theta \, d\theta + \int_0^{\frac{\pi}{2}} 15 \, d\theta$$

$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$

$$= \frac{15}{2} \int_0^{\frac{\pi}{2}} 1 + \cos(2\theta) \, d\theta + \int_0^{\frac{\pi}{2}} 15 \, d\theta$$

$$= \frac{15}{2} \left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}} + \left[15\theta\right]_0^{\frac{\pi}{2}}$$

$$U'_{1\to2} = \frac{15\pi}{4} + \frac{15\pi}{2} = \frac{45\pi}{4} = 11.25\pi$$

$\pi = 3.14$

$$U'_{1\to2} = 35.3 \text{ J}$$

BRINGING IT ALL TOGETHER

$$U'_{1\to2} = \Delta T + \Delta V_g$$

$$35.3 = 0.667 v^2 + 11.77 \quad \text{joules}$$

$$\Rightarrow v_A^2 = 35.28 \, (m/s)^2$$

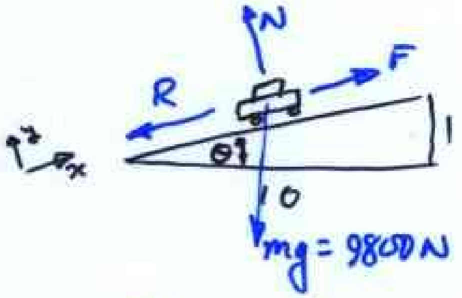
$$\Rightarrow \boxed{v_A = 5.94 \text{ m/s}}$$

Car, Mass = 1000kg. Engine Produces 21kW climbs a hill

G10

(a) when slope is 10%, steady speed of $15ms^{-1}$, find Resistance to motion

(b) Slope flattens out to 4%, & speed still at initial $15ms^{-1}$ find initial acceln (assume resistance unchanged).



R is resistance to motion

Power = 21 000 W

Power is $\vec{F} \cdot \vec{v}$

$$\Rightarrow (F)(15) = 21000$$

$$\& \quad F = \frac{21000}{15} = 1400N$$

(a) Resolve mg into component DOWN HILL & NORMAL TO ROAD. DOWN-HILL $\Rightarrow (1000)(9.8)(\sin\theta) \approx 980N$

No ACCLN

$$\Rightarrow \sum F_x = 0 \Rightarrow F - R - (mg)_x = 0$$

$$R = F - (mg)_x = 1400 - 980 = 420N$$

(b) Slope of 4% $\Rightarrow \sin\theta \approx 0.04$

$$\therefore (mg)_x = (9800)(0.04) = 392\,N$$

Tractive force is still 1400N (Power & velocity unchanged)

Resistance still 420N (we're told this)

$$\sum F_x = ma_x$$

$$\Rightarrow F - R - (mg)_x = (ma)_x$$

$$\frac{(1400 - 420 - 392)}{1000} = a_x \Rightarrow a_x = 0.59\,ms^{-2}$$

m

NOTE
if power stays const, over time $v\uparrow \Rightarrow F\downarrow \Rightarrow a\downarrow$ also.