PLANE KINETICS OF RIGID BODIES

EXTERNAL FORCES & RESULTING TRANSLATIONAL and ROTATIONAL MOTIONS

KINEMATICS V. IMPORTANT

$$\boxed{\sum \vec{F} = m\vec{a}_G}$$

$\vec{F}$ = force

m = mass

$\vec{a}_G$ = acceleration of mass center

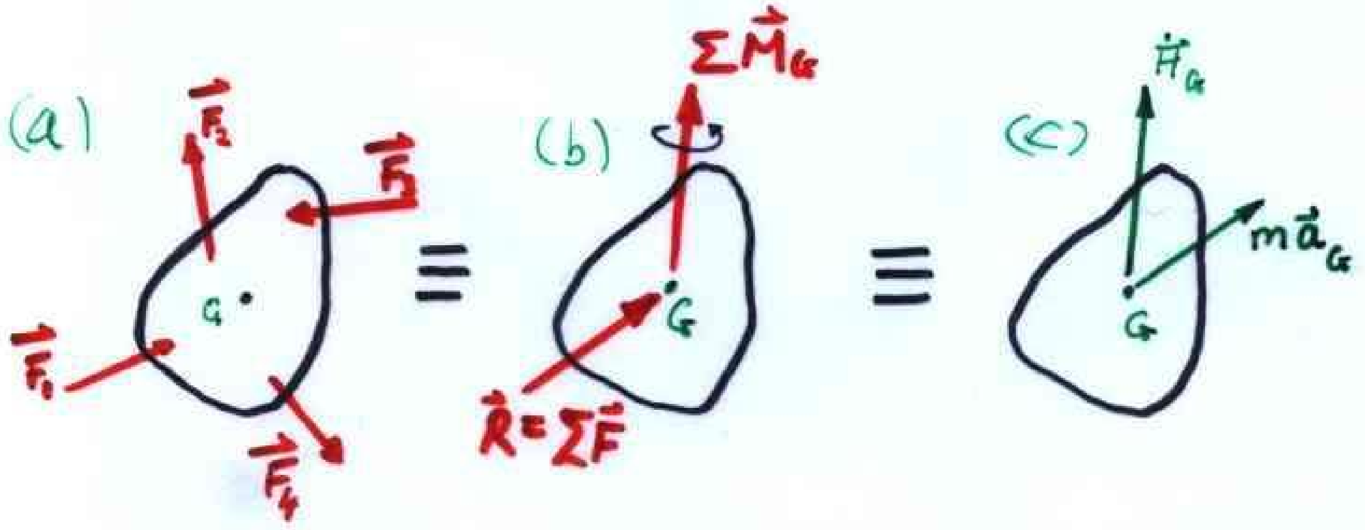
$$\boxed{\sum \vec{M}_G = \dot{\vec{H}}_G}$$

$\vec{M}_G$ = moment about center of mass

$\dot{\vec{H}}_G$ = rate of change of angular momentum about c. of. mass.

NOTE: BOTH ARE VECTOR SUMS.

THESE ARE GENERAL EQUATIONS ⇒ WIDELY USEFUL



F.B.D.

EQUIV FORCE/COUPLE at G

KINETIC DIAGRAM

Simplification: PLANE MOTION

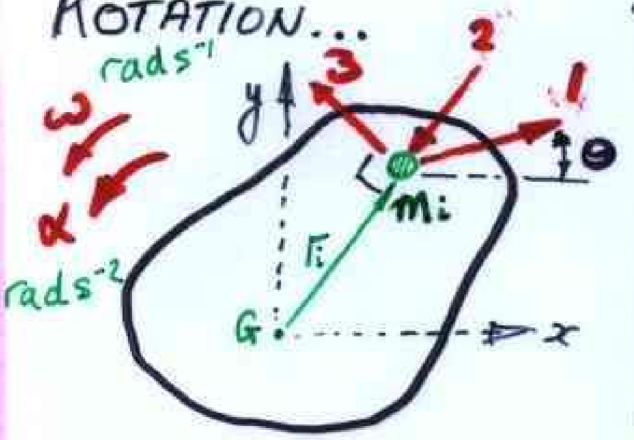
ALL FORCES, MOVEMENT (TRANSLATION & ROTATION) IN 1 PLANE.

ALL MOMENTS $\perp$ to this PLANE

↳ CAN USE SCALAR NOTATION FOR MOMENT, ANGULAR ACCLN, ETC.,

STILL: $\boxed{\sum \vec{F} = m\vec{a}_G}$ ... TRANSLATION

ROTATION...



3 COMPONENTS OF ACCEL.

⇒ 3 comp. of force

1. $m_i \vec{a}_G$
2. $m_i r_i \omega^2$ ... toward G
3. $m_i r_i \alpha$ ... $\perp$ to $\vec{r}_i$

LOOK AT MOMENT OF EACH COMPONENT: **ABOUT G**

1. $M_G^1 = m_i a_G \sin(\theta) x_i - m_i a_G \cos(\theta) y_i$
2. $M_G^2 = 0$ (DIRXN of ω DOES not matter)
3. $M_G^3 = (m_i r_i \alpha)(r_i) = m_i r_i^2 \alpha$

ADD 1. 2. & 3. ... then sum all the m_i that make up THE BODY:

$$\sum M_G = a_G \sin(\theta) \underbrace{\sum m_i x_i}_{\text{ZERO}} - a_G \cos(\theta) \underbrace{\sum m_i y_i}_{\text{ZERO}} + \alpha \sum m_i r_i^2$$

BECAUSE ORIGIN @ G

$$\sum M_G = \alpha \sum m_i r_i^2 \Rightarrow \boxed{\sum M_G = \alpha \int r^2 dm = \alpha I_G}$$

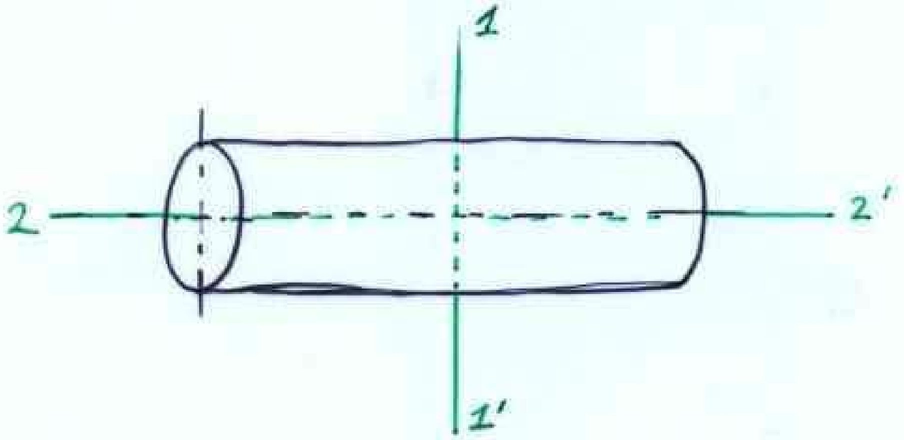
CONST OVER BODY

Mass Moment of INERTIA:

$$\boxed{I = \int r^2 \, dm}$$

N.B. relative to a given axis

I represents a body's RESISTANCE TO ROTATION ABOUT A PARTICULAR AXIS.



$I_{11'} > I_{22'}$ ... DIFFERENT AXIS $\Rightarrow$ DIFFERENT I in general.

2-D PLANE MOTION

$\Rightarrow$ all AXES PARALLEL & LOOK LIKE POINTS.

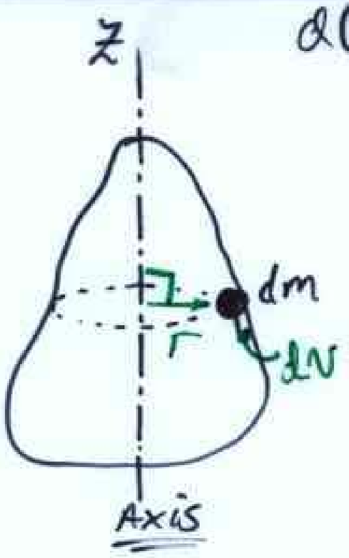
2-D ONE AXIS SPECIAL ... THROUGH CENTER of MASS

DENOTE $\boxed{I_G = \int r^2 \, dm}$

(in 3D LOTS of axes go throug G, BUT we won't look at this in the current course)

for I_G, axis is PERPENDICULAR TO PLANE OF MOTION. (2D problems)

OFTEN DENSITY (ρ) IS CONSTANT.



$dm = \rho\, dv$ (dv is VOLUME of dm)

$I = \int r^2 dm = \int r^2 \rho\, dv$

ρ const

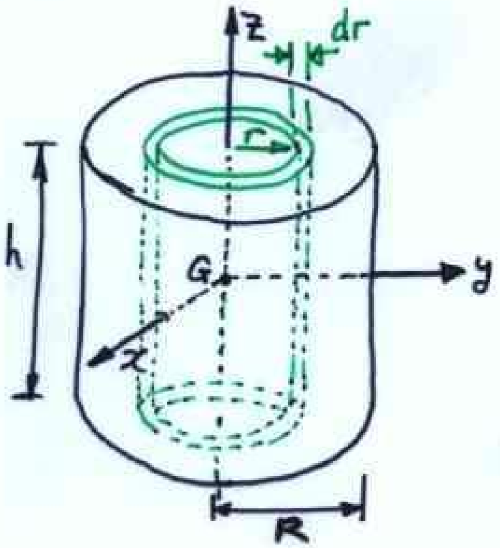
$\Rightarrow I = \rho \int r^2 dv$

How to evaluate $\int r^2 dv$?

Depends on geometry ... some examples

SOLID CYLINDER: radius R, height h

density ρ



CONSIDER ELEMENTAL SHELL

radius r

wall thickness dr

$\Rightarrow$ VOL $= (2\pi r)(dr)(h) = dv$

$I_G = \rho \int r^2 dv$

$= \rho \int_0^R 2\pi h r^3 dr$

$= \rho \frac{\pi}{2} R^4 h$

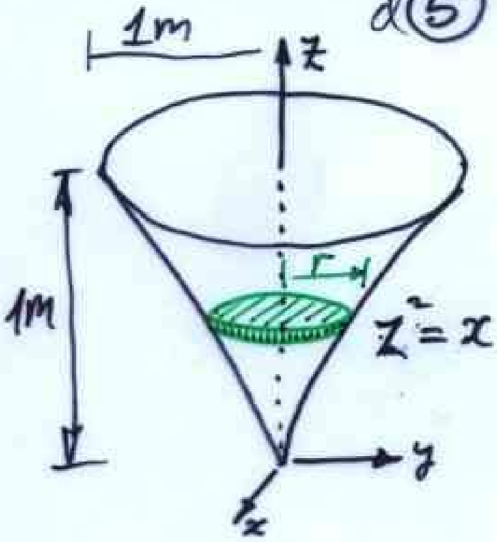
NOTE VOL $= \pi R^2 h$

$\therefore I_G = \frac{mR^2}{2}$

d5

FIND I about z axis

$\rho = 5000\ kg/m^3$



USE INFINITESSIMAL DISK ELEMENT

$$mass = \rho\, dV = \rho(\pi x^2\, dz)$$

I_G of Disk $= \frac{1}{2} m R^2$
(from last e.g.)

$$\Rightarrow dI_z = \frac{1}{2}\, dm\, x^2$$
$$= \frac{1}{2}\left[\rho \pi x^2\, dz\right] x^2$$

$x = z^2$; $I_z = \int_0^1 dI_z$

$$= \frac{\rho \pi}{2} \int_0^1 z^8\, dz$$
$$= 873\ kg\, m^2$$

ought to use "r" not x or y

Parallel Axis Theorem:

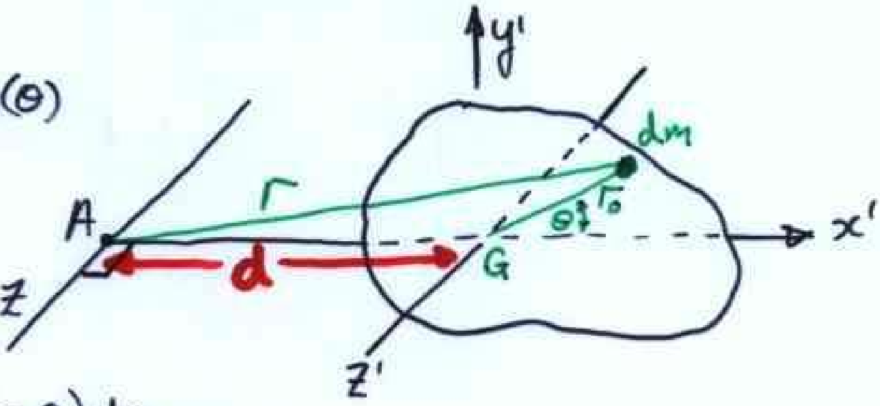
If I_G is known, then I for another parallel axis is

$$\boxed{I = I_G + md^2}$$

d is $\perp$ distance between the axes

$r^2 = r_0^2 + d^2 + 2r_0 d \cos(\theta)$

$I = \int r^2 \, dm$



$$I = \int (r_0^2 + d^2 + 2r_0 d \cos\theta)\, dm$$

$$= \int r_0^2 \, dm + d^2 \int dm + 2d \underbrace{\int r_0 \cos\theta \, dm}_{\text{ZERO because of definition of center of mass.}}$$

$$\Downarrow$$

$$\therefore \; I = I_G + md^2$$

Radius of Gyration k

$$\boxed{k = \sqrt{\frac{I}{m}}} \quad \text{i.e.} \quad \boxed{I = k^2 m}$$

if you are told k, m, then you know "I."

N.B. k is "about an axis", just like I

Body will have different k for different axes... in general.

COMPOSITE BODIES

moment of INERTIA OF A COMPOSITE BODY ABOUT AN AXIS IS THE ALGEBRAIC sum OF THE MOI's of THE COMPONENTS (PARTS) ABOUT THAT AXIS

PARALLEL AXIS THEOREM USEFUL.

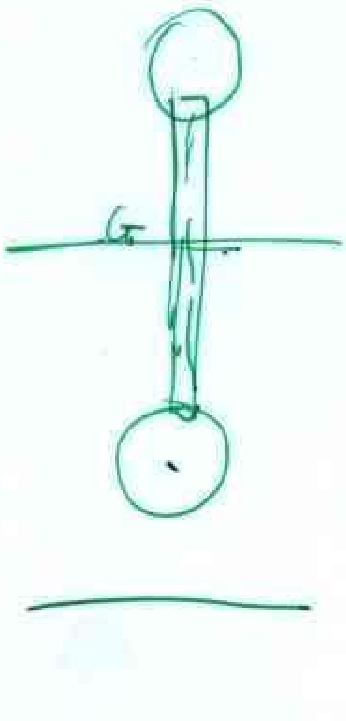
$$I = \sum (I_G + md^2)$$ for composite body

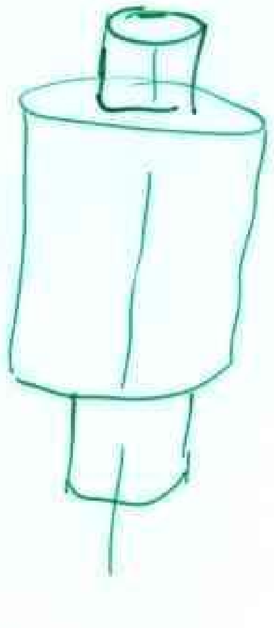
ALSO, YOU CAN SUBTRACT BODIES

e.g. CYLINDER WITH EMBEDDED SPHERICAL CAVITY,

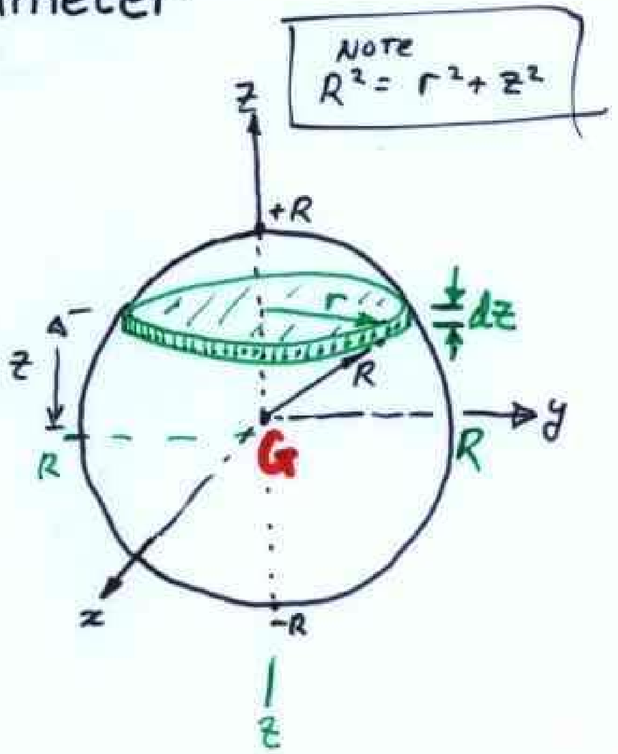
OR HOLLOW TUBE.

SUBTRACT M.O.I. of "MISSING" BITS





EXAMPLES: FIND MOI of a SOLID Sphere about a diameter



FIND I_{zz}

USE DISK AS ELEMENT

$$dI_{zz} = \frac{1}{2}(dm)r^2$$

$$= \frac{\pi\rho}{2}(R^2 - z^2)^2 dz$$

$$I_{zz} = \int dI_{zz}$$

$$= \int_{-R}^{+R} \frac{\pi\rho}{2}(R^2 - z^2)^2 dz$$

$$= \frac{\pi\rho}{2}\int_{-R}^{+R}(R^4 - 2R^2z^2 + z^4)\, dz$$

$$= \frac{\pi\rho}{2}\left[R^4 z - \frac{2R^2z^3}{3} + \frac{z^5}{5}\right]_{+R}^{-R}$$

$$= \frac{\pi\rho}{2}\left(\left[R^5 - \frac{2R^5}{3} + \frac{R^5}{5}\right] - \left[-R^5 + \frac{2R^5}{3} - \frac{R^5}{5}\right]\right)$$

$$= \frac{\pi\rho}{2}(2)\left(R^5 - \frac{2R^5}{3} + \frac{R^5}{5}\right)$$

$$= \frac{8\pi\rho R^5}{15}$$

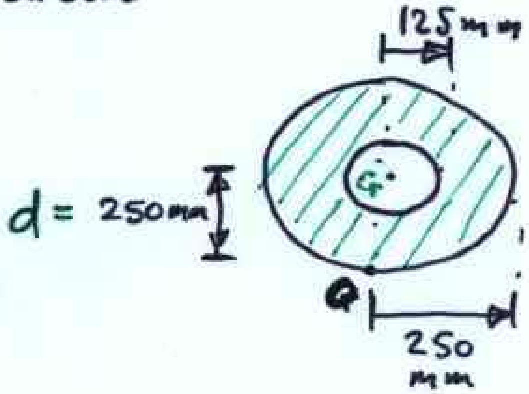
NOTE $m = \rho\frac{4}{3}\pi r^3$ for sphere

$$\therefore I_{zz} = \frac{2}{5}mR^2$$

$$k = \sqrt{\frac{I}{m}} = \sqrt{\frac{2}{5}}\,R$$ Radius of gyration

FIND M.o.I of 10mm thick circular plate about axis $\perp$ to page thru O

PLATE HAS HOLE AS SHOWN.



Composite body ...

intact disk $I_G = \frac{1}{2} m r^2$

$\parallel$ axis thm: $I_o = I_g + m d^2 = \frac{1}{2} m r^2 + m d^2$

ρ of steel

$m_{DISK} = \rho_{DISK} V_{DISK} = 8000\ kg/m^3 \left[\pi (0.25)^2 (0.01)\right]$

$= 15.71\ kg$

$(I_{DISK})_o = m_D \left(\frac{1}{2} r_D^2 + d^2\right)$

$= 1.473\ kg\ m^2$

Now ... look @ HOLE

$m_{HOLE} = \rho_H V_H = 8000\ kg/m^3 \left[\pi (0.125)^2 (0.01)\right]$

$= 3.93\ kg$

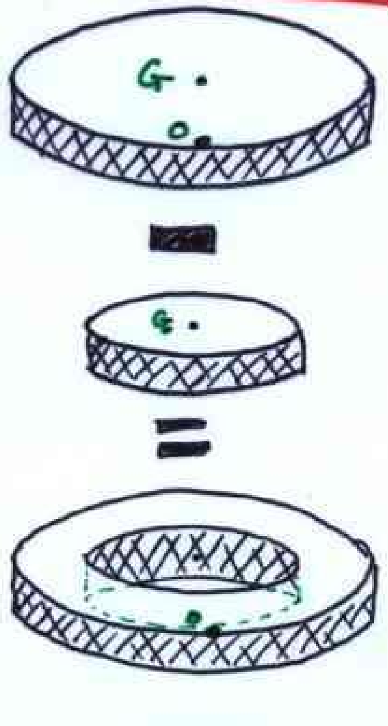
$(I_H)_o = m_H \left(\frac{1}{2} r_H^2 + d^2\right)$

$= 0.276\ kg m^2$

$(I_{PLATE})_o = (I_{DISK})_o - (I_{HOLE})_o$

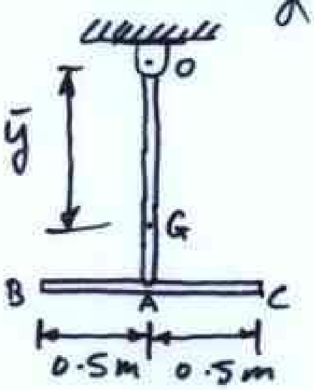
$= 1.473 - 0.276$

$= 1.20\ kg\ m^2$

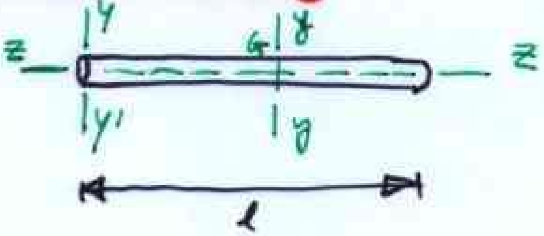


PENDULUM MADE of 2 slender RODS 50 N weight each.

find M.o.I about axis through O and through G.



FOR a ROD



$I_{zz} = 0$

$I_{yy} = \frac{1}{12} m l^2$

$I_{yy'} = \frac{1}{3} m l^2$

FIND I_o ...

mass

$$(I_{OA})_o = \frac{1}{3}\left(\frac{50}{g}\right)(1.0)^2 = \underline{1.699}\ kg\,m^2$$ ROD OA

$$(I_{BC})_o = \frac{1}{12} m l^2 + m d^2$$ ROD BC

$$= \frac{1}{12}\left(\frac{50}{g}\right)(1.0)^2 + \left(\frac{50}{g}\right)(1.0)^2$$

$$= \underline{5.522}\ kg\,m^2$$

$$\therefore I_o = (I_{OA})_o + (I_{BC})_o = \underline{\underline{7.22}}\ kg\,m^2$$ COMBO

To get I_G, need to find $\bar{y}$

Def $$\bar{y} = \frac{\Sigma \tilde{y} m}{\Sigma m} = \frac{0.5\left(\frac{50}{g}\right) + 1.0\left(\frac{50}{g}\right)}{\frac{50}{g} + \frac{50}{g}} = 0.75\ m$$

$\boxed{\frac{50}{g} = \text{mass}}$

PARALLEL AXIS thm

$$I_o = I_g + m d^2 \iff I_G = I_o - m d^2 \ ; \ d = \bar{y}$$

$$I_G = 7.22 - \left(\frac{100}{g}\right)(0.75)^2$$

$$\boxed{I_g = 1.486\ kg\,m^2}$$

$\boxed{g = 9.81\ m s^{-2}}$