# General Trigonometric Relationships

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin \theta/2 = \sqrt{1/2\,(1-\cos\theta)}$$

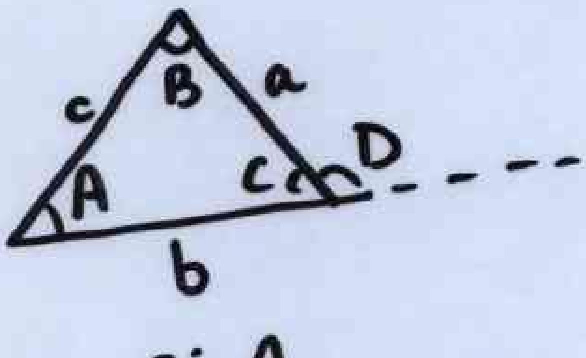
$$\cos \theta/2 = \sqrt{1/2\,(1+\cos\theta)}$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$$

$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$$




$$\frac{a}{b} = \frac{\sin A}{\sin B}$$ ...Sine Law

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Cosine Law

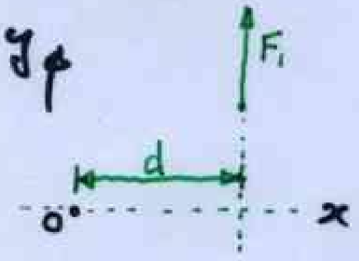
$$c^2 = a^2 + b^2 + 2ab\cos D$$

FORCE

↳ LINEAR ACCELERATION $\vec{F} = m\vec{a}$

↳ ROTATION ABOUT AN AXIS MOMENT $\vec{M} = \vec{r} \times \vec{F}$

2-D simple: AXIS IS A POINT "o"




$\vec{M}_o = +F_1 d$

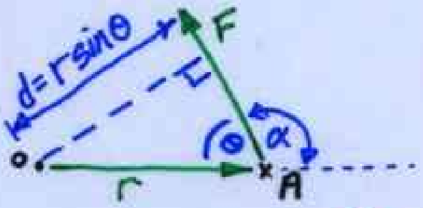

GET SIGN FROM RIGHT HAND RULE

IN 2-D MOMENT EITHER "UP" OR "DOWN"

$+$ $\qquad$ $-$

FOR 3-D CASE VECTOR APPROACH HELPS

$\vec{M} = \vec{r} \times \vec{F}$ (Note order!)




if $\vec{r} = 2\hat{\imath}$ & $\vec{F} = -\hat{\imath} + 2\hat{\jmath}$ then

$$\vec{M} = \begin{Vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 0 & 0 \\ -1 & 2 & 0 \end{Vmatrix} = \hat{\imath}(0) - \hat{\jmath}(0) + \hat{k}(2 \cdot 2 + 1 \cdot 0)$$

DETERMINANT

$$= 4\hat{k}$$


RIGHT HAND RULE CONFIRMS SIGN/DIRXN CORRECT.

VARIGNON'S THEOREM

"the MOMENT OF A FORCE ABOUT AN AXIS IS EQUAL TO THE SUM OF THE MOMENTS OF ITS COMPONENTS ABOUT THAT AXIS"

PROOF: $M_o = \vec{r} \times \vec{F}$ ; $\vec{F} = \vec{P} + \vec{Q}$ COMPONENTS

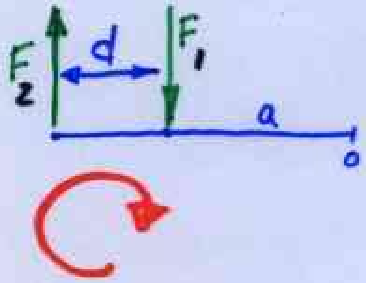
$\Rightarrow M_o = \vec{r} \times (\vec{P} + \vec{Q})$

CROSS PRODUCT is DISTRIBUTIVE SO...

$M_o = \vec{r} \times \vec{P} + \vec{r} \times \vec{Q}$ i.e. SUM OF MOMENTS.

## COUPLE:

CONSIDER 2 EQUAL AND OPPOSITE FORCES. NETT FORCE $= \vec{0}$.



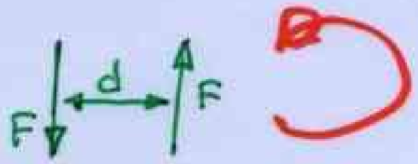

WHAT IS MOMENT?

$$M_o = +\overset{1}{F}.a - \overset{2}{F}(a+d)$$
$$= -Fd$$

i.e. "a" DOES NOT MATTER.

MOMENT OF COUPLE SAME ABOUT ALL POINTS

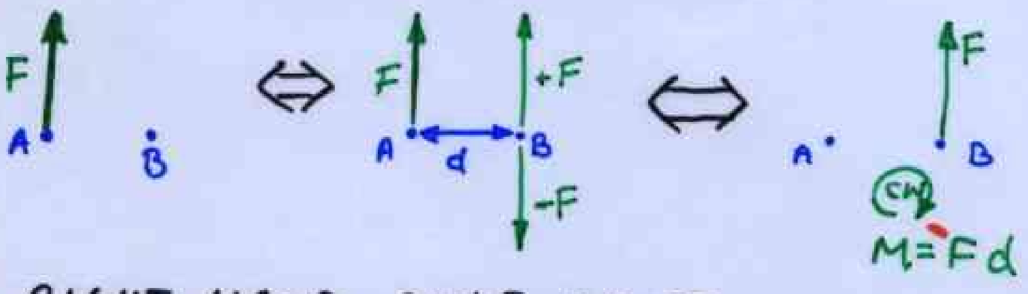
NOTE SIGN... if WE HAD




then $\vec{M} = +Fd$ (RIGHT HAND RULE, AGAIN)

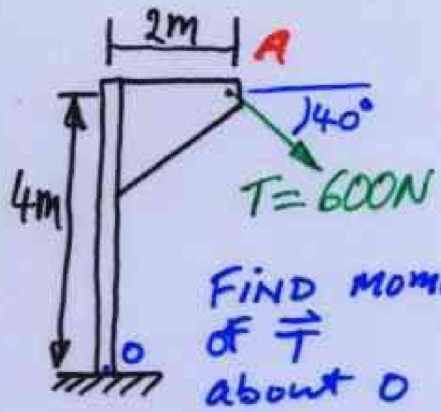
## FORCE - COUPLE SYSTEMS:

YOU CAN REPLACE A FORCE BY AN EQUAL PARALLEL FORCE PLUS A MOMENT:




RIGHT HAND RULE GIVES DIRECTION OF COUPLE (clockwise/NEGATIVE HERE) b3

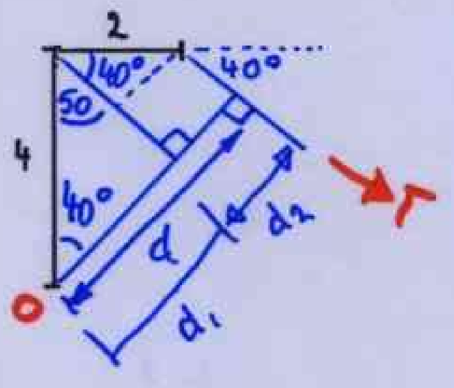
# SAMPLE PROBLEM 2/5




FIND MOMENT of $\vec{T}$ about O

5 WAYS TO SOLVE:

## I/ GEOMETRY



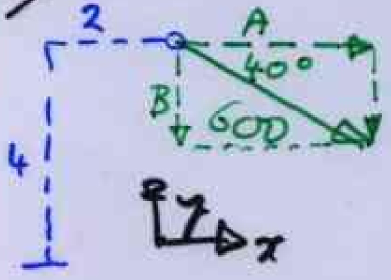

$d_1 = 4\cos(40°)$

$d_2 = 2\sin(40°)$

$d = d_1 + d_2 = 4.35\text{m}$

$\Rightarrow \vec{M}_o = \underline{2610\,\text{Nm}}$ CLOCKWISE. OR $\underline{-2610\,\text{Nm}}$ (Right HAND RULE)

## II/ COMPONENTS:



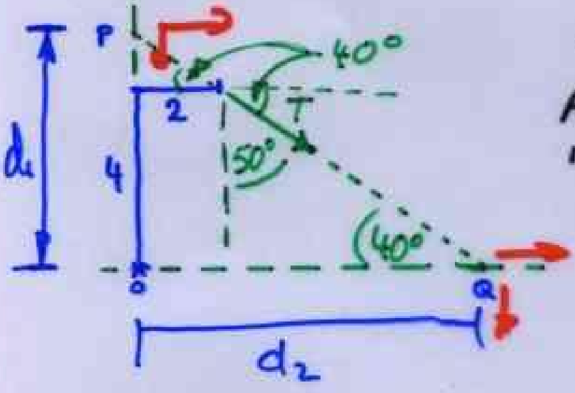

$\vec{T} = (600)(\cos 40°)\hat{\imath} + (600)(\sin 40°)\hat{\jmath}$

$\vec{T} = \underset{A}{460}\,\hat{\imath} \pm \underset{B}{386}\,\hat{\jmath}$

TOTAL MOMENT IS NOW:

$(460)(4) + (386)(2) = \underline{2610\,\text{Nm}}$ (DIRXN AS BEFORE)

## III TRANSMISSIBILITY + COMPONENTS




FORCE $\vec{T}$ CAN BE MOVED ALONG LINE OF ACTION to P OR Q

AT P ONLY $\hat{\imath}$ COMPONENT HAS MOMENT

" Q " $\hat{\jmath}$ " " "

At P: $\vec{M} = (460)(d_1)$

$= (460)(2\cdot\tan(40°) + 4)$

$= \underline{2610\,\text{Nm}}$

At Q: $\vec{M} = 460\,d_2 = \cancel{460}\,386\,(2 + 4\tan(50°))$

$= \underline{2610\,\text{Nm}}$

## IV CROSS PRODUCT:

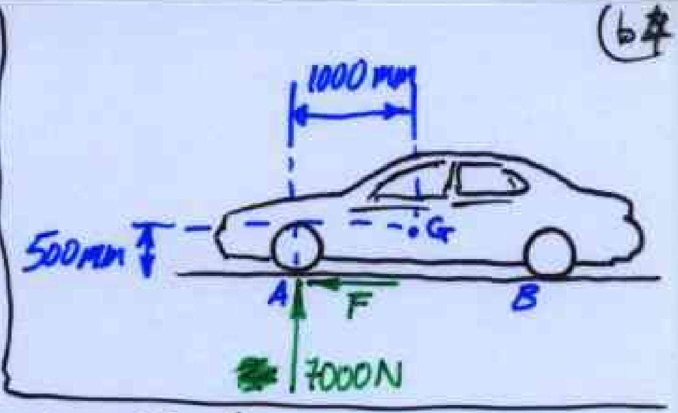
$\vec{M}_o = \vec{r} \times \vec{F} = (2\hat{\imath} + 4\hat{\jmath}) \times (\hat{\imath}\cos 40° \cancel{+} - \hat{\jmath}\sin 40°)\,600$

$\vec{M}_o = -\underline{2610\,\hat{k}}$ N·m NOTE SIGN

Q.2/71

2D Problem

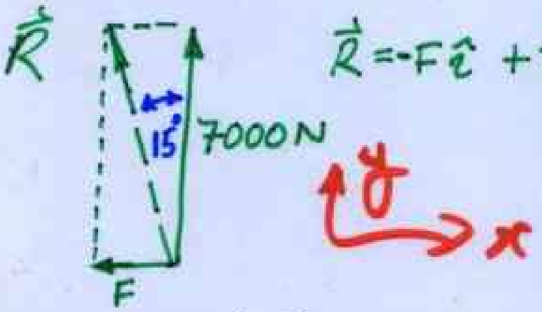
FRONT WHEELS EXPERIENCE COMBINED REACTION of 7000N plus friction $\vec{F}$. Both from ROAD.




Resultant makes 15° angle to vertical

find EQUIVALENT FORCE-COUPLE system @ G

I/ Easiest way




$\vec{R} = -F\hat{i} + 7000\hat{j}$

BASIC TRIG:

$\|\vec{R}\| = 7000/\cos(15°)$

$= 7000/0.966 = \underline{7247\ N}$

$\|\vec{F}\| = (7000)(\tan 15°)$

$= (7000)(0.268) = \underline{1876\ N}$

moment of $\vec{R}$ about G = $\Sigma$ moments of components

$M_G = (-7000)(1.0) + (-1876)(0.5)$

$= -7000 - 938 = \underline{-7938\ Nm}$. i.e. 7938 Nm $\underline{CW}$

So force-couple @ G = $\underbrace{-1876\hat{i} + 7000\hat{j}}_{\text{force}}$ AND $\underbrace{-7938\,Nm}_{\text{couple}}$

II/ Calculate moment Differently:

$M = Rd$

$R = 7247N$

$d_2 = 1.0/\cos(15°) = 1.0/0.966 = \underline{1.035}$

$d_1 = (L)\sin(15°)$

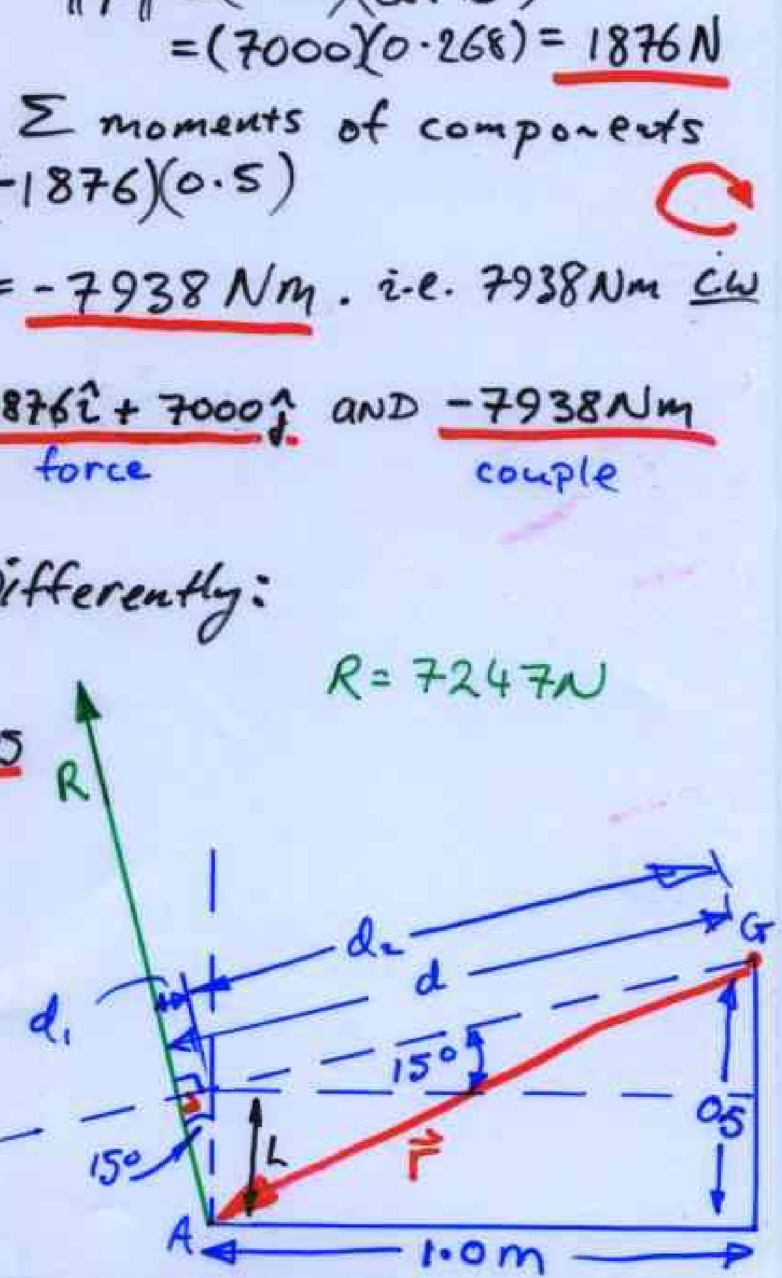
$L = 0.5 - (1.0)(\tan 15°)$

$d_1 = (0.5 - 0.268)(0.259) = \underline{0.06}$

$d = 1.035 + 0.06 = \underline{1.095}$

$M = (1.095)(7247)$

$= 7935\ Nm$ $\boxed{C.W.}$



# III/ CROSS PRODUCT:

from Before : $\vec{R} = -1876\,\hat{\imath} + 7000\,\hat{\jmath}$

$\vec{r}$ ... vector from G to A is $-1.0\,\hat{\imath} - 0.5\,\hat{\jmath}$

then $\vec{r} \times \vec{R}$ is $\text{Det}\begin{pmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1.0 & -0.5 & 0 \\ -1876 & 7000 & 0 \end{pmatrix}$

$= \hat{\imath}\left((-0.5)(0) - (7000)(0)\right)$

$-\hat{\jmath}\left((-1.0)(0) - (-1876)(0)\right)$

$+\hat{k}\left((-1.0)(7000) - (-1876)(-0.5)\right)$

$= \hat{k}(-7000 - 938)$

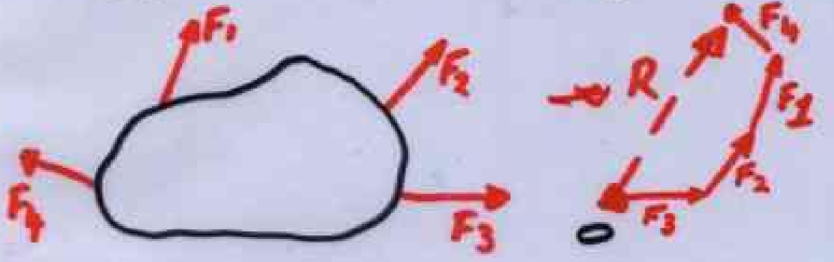
$\vec{M}_k = -7938\,\hat{k}$ same as before including sign

# RESULTANTS:

concepts already COVERED: FORCE, MOMENT, COUPLE

Now: we want to describe effect of a system of forces & couples

FOR multiple forces, we can use force polygon (Like triangle rule



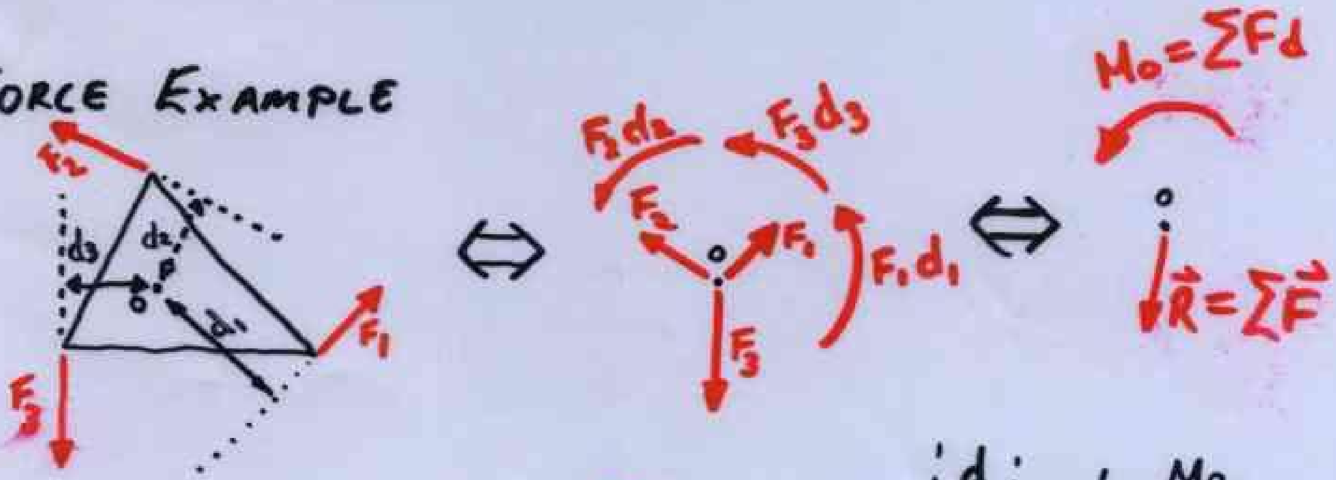

$\vec{R}$ is the RESULTANT.

gives magnitude AND DIRECTION

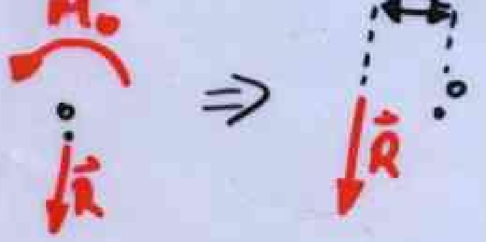
ORDER OF ADDITION DOES <u>NOT</u> MATTER.

TO GET LINE-OF-ACTION WE MUST MOVE FORCES to A CONVENIENT REFERENCE POINT USING FORCE COUPLE SYSTEMS.

3 FORCE EXAMPLE




FINALLY, can eliminate couple by moving force:

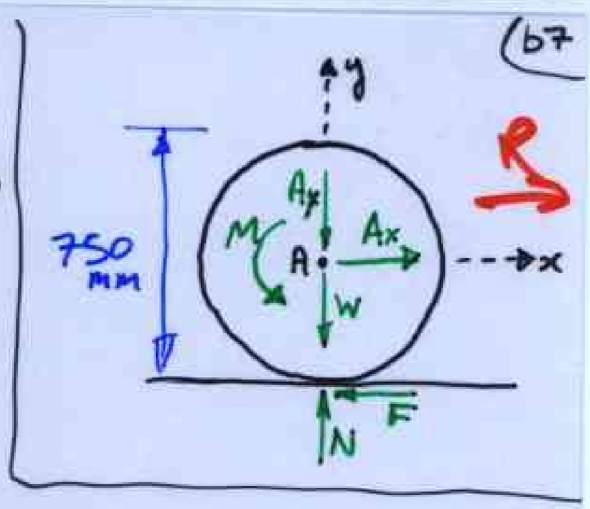



$d = \frac{M_o}{R}$

Choose pt. "o" carefully to make life easier

## PROBLEM 2/89

Rolling Rear wheel of car.
FRONT WHEEL DRIVE & Accelerating to RIGHT.




Axle Forces: $A_x = 240N \Rightarrow \underline{+240\hat{\imath}}$

$A_y = 2000N \Rightarrow \underline{-2000\hat{\jmath}}$

Road friction: $F = 160N \Rightarrow \underline{-160\hat{\imath}}$

" Support: $N = 2400N \Rightarrow \underline{+2400\hat{\jmath}}$

WHEEL weight: $W = 400N \Rightarrow \underline{-400\hat{\jmath}}$ $M = \underline{3\,Nm}$ Bearing friction

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Sum to get nett force on wheel:

$$\hat{\imath}(240 - 160) + \hat{\jmath}(-2000 + 2400 - 400)$$

$\underline{\vec{R} = 80\hat{\imath}}$ ... (comment!) N

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Moment about A:

$\vec{M} = +3\,Nm$

all of $\vec{A}_x, \vec{A}_y, \vec{W}, \vec{N}$ have $\underline{\text{ZERO}}$ moment about A

leaves $\vec{F}$: $M_F = -F(0.75/2)$ Nm

$= -(160)(0.75)/2 = -60\,Nm$

total $\vec{M}_A = 3 - 60 = \underline{-57\,Nm}$

find posn of resultant ...

moment arm ... $57/80 = \underline{0.712\,m}$ above A

By inspection

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Alternatively: $\vec{r} \times \vec{R} = -57\,Nm$ $\vec{r} = x\hat{\imath} + y\hat{\jmath} + 0\hat{k}$ UNKNOWN

$$\Rightarrow \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ x & y & 0 \\ 80 & 0 & 0 \end{vmatrix} = -57\hat{k}$$

$$\Rightarrow \hat{\imath}(0) + \hat{\jmath}(0) + \hat{k}\big((x)(0) - 80y\big) = -57\hat{k}$$


$$-80y = -57$$

$y = 57/80 = \underline{+0.712\,m}$ Again

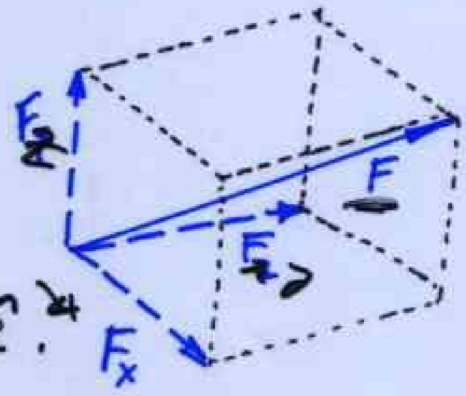
# 3-Dimensional Cases: FORCE



Important to be careful & to use components.
DIFFICULT TO DRAW.

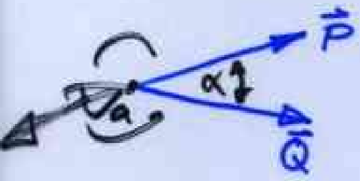
$\vec{F} = \|F\| (\ell \hat{\imath} + m \hat{\jmath} + n \hat{k})$ where $\ell^2 + m^2 + n^2 = 1$

$\equiv F_x \hat{\imath} + F_y \hat{\jmath} + F_z \hat{k}$




DOT PRODUCT:

$\vec{P} \cdot \vec{Q} = \|P\| \, \|Q\| \cos\alpha$




Note VECTORS ORIGINATE at same point for finding $\alpha$.

Also note: $F_x = \vec{F} \cdot \hat{\imath}$, $F_y = \vec{F} \cdot \hat{\jmath}$, $F_z = \vec{F} \cdot \hat{k}$ Projection ONTO AXES

ANGLE BETWEEN VECTORS:

$$\alpha = \cos^{-1}\left(\frac{\vec{P} \cdot \vec{Q}}{\|P\| \, \|Q\|}\right)$$ ... inverse cosine

if $\vec{P} \cdot \vec{Q} = 0 \Rightarrow \vec{P} \perp \vec{Q}$ (assuming $\|P\| \neq 0$ & $\|Q\| \neq 0$)

$\vec{F}_1 \cdot \vec{F}_2$

$F_{1x} F_{2x} + F_{1y} F_{2y} + F_{1z} F_{2z}$

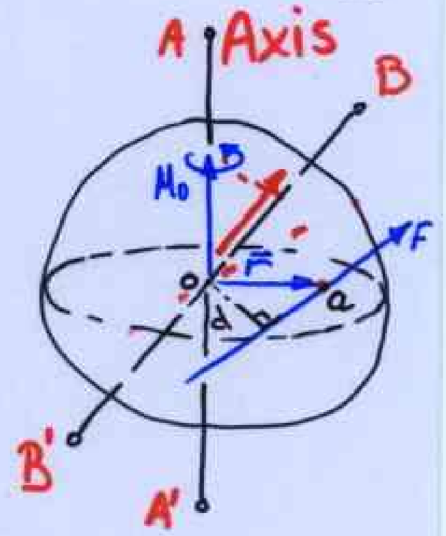
# 3-D : moment & couple



Sometimes just like 2-D case
Force $\vec{F}$ & point O define a plane
moment of $\vec{F}$ about an <u>AXIS</u> A-A'
normal to this plane is

$$\boxed{\vec{M}_o = \vec{r} \times \vec{F}}$$

same as before.




If we want moment about Axis B-B', need to do a bit more work.

note: if B-B' was parallel to $\vec{F}$, moment would be <u>ZERO</u>.

$$\boxed{\vec{M}_B = \left( (\vec{r} \times \vec{F}) \cdot \vec{n} \right) \vec{n}}$$

WHERE $\vec{n}$ is a <u>unit</u> <u>VECTOR</u> along axis B-B'

To UNDERSTAND THIS ... BREAK INTO PIECES

$(\vec{r} \times \vec{F}) = \vec{M}_0$ as before.

1.0

So $(\vec{r} \times \vec{F}) \cdot \vec{n} \equiv \vec{M}_0 \cdot \vec{n} = \|M_0\| \, \|\vec{n}\| \cos\theta$

$\|n\| = 1$ so $\Rightarrow$ <u>$\|M_0\| \cos(\theta)$</u>

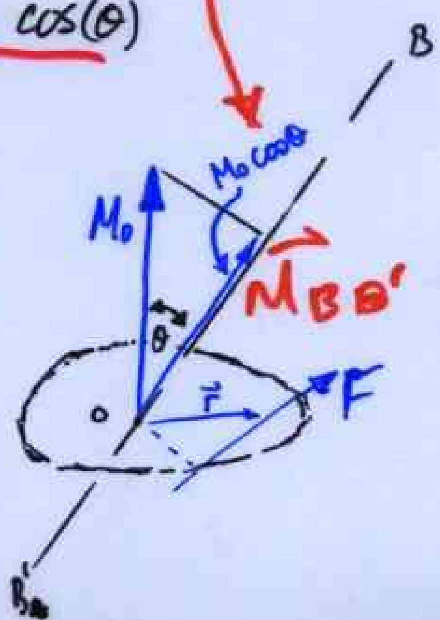
NOTE THIS is A <u>SCALAR</u>.

last bit ... mult by $\vec{n}$

So $\vec{M}_B$ is VECTOR // to $\vec{n}$

with magnitude of $\vec{M}_0$

Projected onto $\vec{n}$



# 3-D moment couple ctd.



VARIGNON'S THEOREM IN 3D

$$\vec{M}_o = \Sigma(\vec{r} \times \vec{F}) = \vec{r} \times \vec{R} \quad \text{where } \vec{R} = \Sigma \vec{F}$$

Couples in 3-D same as 2D really

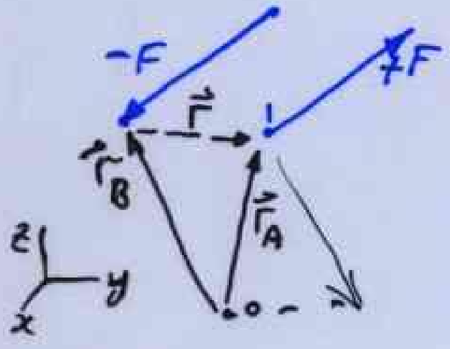
$$M = \vec{r} \times F$$

why?

$$\Rightarrow \vec{M}_o = +\vec{r}_A \times \vec{F} - \vec{r}_B \times \vec{F}$$

$$\vec{M} = (\vec{r}_A - \vec{r}_B) \times \vec{F}$$

$$\vec{M} = \vec{r} \times \vec{F}$$



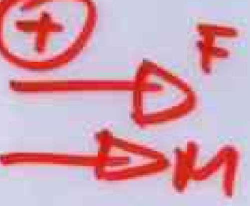

RESULTANTS:

$$\underset{\text{forces}}{\vec{R} = \Sigma \vec{F}} \quad \underline{\text{AND}} \quad \underset{\text{couples}}{\vec{M} = \Sigma \vec{M}} \quad \text{like 2D.}$$

"WRENCH RESULTANT" when resultant force $\vec{F}$ is PARALLEL TO THE RESultant couple.

Positive when $\vec{F}$ & $\vec{M}$ are aligned

+

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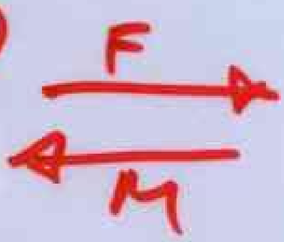
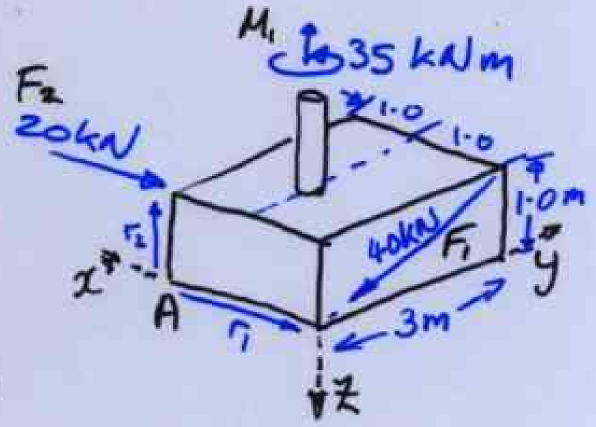


Problem 2/143 (b11

REPLACE 2 forces + couple by a force-couple sys @ A.




Step

1) Express forces in components. Label forces to aid clarity $\vec{F}_1$ $\vec{F}_2$

$\vec{F}_2$ is easy: $\underline{\vec{F}_2 = -20\,kN\,\hat{\imath}}$ Note sign

$\vec{F}_1$ needs trig: $\theta = \tan^{-1}\left(\frac{1.0}{3.0}\right) = 18.4°$

$F_{1y} = (40\,kN)(\cos(18.4°)) = 37.95$

$F_{1z} = (40\,kN)(\sin(18.4°)) = 12.65\,kN$

so $\underline{F_1 = -37.95\,\hat{\jmath} + 12.65\,\hat{k}}$ kN

To get $\vec{R} = \vec{F}_1 + \vec{F}_2$ ... ADD $\Rightarrow$ $\underline{\vec{R} = -20\hat{\imath} - 37.95\hat{\jmath} + 12.65\hat{k}}$ kN

Now to get couple of resultant about A.

we have $\vec{M}_1$ already: $\vec{M}_1 = -35\,\hat{k}$ kNm sign

moments due to $\vec{F}_1$ & $\vec{F}_2$ ...

$$\vec{r}_1 \times \vec{F}_1 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -2 & 0 & 0 \\ 0 & -37.95 & +12.65 \end{vmatrix} \qquad \vec{r}_2 \times \vec{F}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 0 & -1 \\ -20 & 0 & 0 \end{vmatrix}$$

$$= -\hat{\jmath}(-2)(12.65) + \hat{k}(-2)(-37.95) \qquad = +20\,\hat{\jmath}$$

Sum 3 contributions (green)

$$\vec{M} = -35\hat{k} + 20\hat{\jmath} + 25.3\hat{\jmath} + 75.9\hat{k}$$

$$= \underline{45.3\,\hat{\jmath} + 40\,\hat{k}} \quad \underline{kN{\cdot}m}$$